(7.1) 
(i) 
\[ H = I - 2 \frac{uu^T}{u^Tu} \]
\[ Hu = (I - 2 \frac{uu^T}{u^Tu})u = u - 2 \frac{uu^Tu}{u^Tu} = u - 2u = -u \]

(ii) 
\[ Hv = (I - 2 \frac{uu^T}{u^Tu})v = v - 2 \frac{uu^Tv}{u^Tu} = v - 0 = v \quad \text{because} \quad v^Tu = 0. \]

implies that \( u^Tv = 0. \)

7.2 
(ii) 
\[ Hv = (I - 2 \frac{uu^T}{u^Tu})v = v - 2 \frac{uu^Tv}{u^Tu} = v - 0 = v \quad \text{because} \quad v^Tu = 0. \]

For an \( m \times n \) matrix \( A \), prove the following results using the SVD of \( A \):
(i) \( \text{rank}(A^T A) = \text{rank}(AA^T) = \text{rank}(A) = \text{rank}(A^T) \).
(ii) \( A^T A \) and \( AA^T \) have the same nonzero eigenvalues.
(iii) If the eigenvectors \( u_1 \) and \( u_2 \) of \( A^T A \) are orthogonal, then \( Au_1 \) and \( Au_2 \) are orthogonal.

-Solution-
Without the loss of generality, we will assume that \( m \geq n \) for all problems of No. 7.19.

**Singular Value Decomposition**
For a matrix \( A \in \mathbb{R}^{m \times n} \), there exist orthogonal matrices \( U \in \mathbb{R}^{m \times m} \) and \( V \in \mathbb{R}^{n \times n} \) such that \( A = U \Sigma V^T \) where \( \Sigma = \text{diag} (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{m \times n} \) and \( \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0 \).

(i) 
From the SVD of \( A \), there are orthogonal matrices \( U \) and \( V \) such that \( A = U \Sigma V^T \). Then we have 
\[ A^T = V \Sigma^T U^T. \]

However, we know \( \Sigma = \text{diag} (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{m \times n} \). This implies \( \Sigma^T = \text{diag} (\sigma_1, \ldots, \sigma_n) \in \mathbb{R}^{n \times n} \).

Also, we know that \( \text{rank}(A) = \text{The number of nonzero singular values} \).

Since the singular values of \( A \) and \( A^T \) are \( \sigma_1, \sigma_2, \ldots, \sigma_n \), the number of nonzero singular values of \( A \) is equal to the number of nonzero singular values of \( A^T \). Therefore, \( \text{rank}(A) = \text{rank}(A^T) \).

Now consider \( AA^T \) and \( A^T A \). Since \( A = U \Sigma V^T \) and \( A^T = V \Sigma^T U^T \), we have 
\[ AA^T = (U \Sigma V^T)(V \Sigma^T U^T) = U(\Sigma \Sigma^T)U^T \quad \text{and} \quad A^T A = (V \Sigma^T U^T)(U \Sigma V^T) = V(\Sigma^T \Sigma)V^T \]
Looking at $\Sigma^T \Sigma$ and $\Sigma \Sigma^T$, we can see

$$\Sigma \Sigma^T = \begin{pmatrix}
\begin{array}{cccc}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & 0 & \\
\ddots & \ddots & \ddots & \\
0 & \cdots & \cdots & \sigma_n
\end{array}
\end{pmatrix}_{m \times n}
$$

and

$$\Sigma \Sigma^T = \begin{pmatrix}
\begin{array}{cccc}
\sigma_1 & 0 & \cdots & 0 \\
0 & \sigma_2 & 0 & \\
\ddots & \ddots & \ddots & \\
0 & \cdots & \cdots & \sigma_n
\end{array}
\end{pmatrix}_{n \times m}$$

Similarly, we can obtain

$$\Sigma^T \Sigma = \begin{pmatrix}
\begin{array}{cccc}
\sigma_1^2 & 0 & \cdots & 0 \\
0 & \sigma_2^2 & 0 & \\
\ddots & \ddots & \ddots & \\
0 & \cdots & \cdots & \sigma_n^2
\end{array}
\end{pmatrix}_{m \times m}$$

From (1) and (2), the singular values of $AA^T$ and $A^T A$ are $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. Therefore, the number of nonzero singular values of $AA^T$ is equal to the number of nonzero singular values of $A^T A$, that is, $\text{rank}(AA^T) = \text{rank}(A^T A)$.

Also, from the facts that the singular values of $A$ are $\sigma_1, \sigma_2, \ldots, \sigma_n$ and that the singular values of $AA^T$ are $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, the number of nonzero singular values of $A$ is equal to the number of nonzero singular values of $AA^T$. Therefore, $\text{rank}(A) = \text{rank}(AA^T)$.

Finally, we showed that

$$\text{rank}(A^T A) = \text{rank}(AA^T) = \text{rank}(A) = \text{rank}(A^T).$$

(ii)

By the Singular Value Decomposition, we have

$$A = U \Sigma V^T \quad \text{where} \quad \Sigma_{m \times n} = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$$

$$A^T = V \Sigma^T U^T \quad \text{where} \quad \Sigma_{n \times m} = \text{diag}(\sigma_1, \sigma_2, \cdots, \sigma_n)$$

By the definition of singular values, for a real matrix $A$, the square roots of the eigenvalues of $A^T A$ are called the singular values of $A$, that is, the eigenvalues of $A^T A$ are $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. Since $AA^T = (A^T)^T A^T$ and the singular values of $A^T$ are $\sigma_1, \sigma_2, \ldots, \sigma_n$, the eigenvalues of $AA^T = (A^T)^T A$ are $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. Therefore, $A^T A$ and $AA^T$ have the same nonzero eigenvalues.
Suppose that the eigenvectors $u_1$ and $u_2$ of $A^T A$ are orthogonal. Then we have

$$
(A^T A)u_1 = \lambda_1 u_1 \quad \cdots (1)
$$
$$
(A^T A)u_2 = \lambda_2 u_2 \quad \cdots (2)
$$
$$
u_1^T u_2 = u_2^T u_1 = 0 \quad \cdots (3)
$$

Then consider $(Au_1)^T (Ay_2)$ and $(Au_2)^T (Au_1)$. Using (2) and (3), we have

$$
(Au_1)^T (Au_2) = u_1 A^T A u_2 = u_1^T (\lambda_2 u_2) = \lambda_2 u_1^T u_2 = 0
$$

Similarly, using (1) and (2), we have

$$
(Au_2)^T (Au_1) = u_2 A^T A u_1 = u_2^T (\lambda_1 u_1) = \lambda_1 u_2^T u_1 = 0
$$

Therefore, $Au_1$ and $Au_2$ are orthogonal.

**7.21**

Let $U$ have orthogonal columns. Then using SVD, prove that

(i) $\|AU\|_2 = \|A\|_2$

(ii) $\|AU\|_F = \|A\|_F$

-Solution-

Again without the loss of generality, we will assume that $m \geq n$ for all problems of No. 7.21.

Since $U$ has orthogonal columns, $U = [u_1 u_2 \cdots u_n]_{m \times n}$, where $u_i$ is the $i^{th}$ column vector expressed as $u_i = (u_{1i}, u_{2i}, \ldots, u_{ni})^T$ and satisfying

$$
u_i^T u_j = \begin{cases} 
0 & \text{if } i \neq j \\
1 & \text{if } i = j
\end{cases}
$$

This implies

$$
U^T U = \begin{pmatrix}
    u_1^T \\
    u_2^T \\
    \vdots \\
    u_n^T
\end{pmatrix}
\begin{pmatrix}
    u_1 & u_2 & \cdots & u_n^T
\end{pmatrix}_{m \times n} = I_{n \times n}
$$

Therefore, $U$ is an orthogonal matrix.

From the Singular Value Decomposition, for a matrix $A$, there exist orthogonal matrices $\hat{U}$ and $\hat{V}$ such that $A = \hat{U} \Sigma \hat{V}^T$, where $\Sigma = diag(\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Now multiplying $A = \hat{U} \Sigma \hat{V}^T$ by $U$, we have

$$
AU = \hat{U} \Sigma \hat{V}^T U
$$

Since the product of orthogonal matrices is an orthogonal matrix, letting $W^T = \hat{V}^T U$, we have

$$
AU = \hat{U} \Sigma W^T \quad \cdots (\ast)
$$

(i)

Since $\|A\|_2 = \sigma_{\max}$, from (\ast) we have $\|AU\|_2 = \sigma_{\max}$. Therefore, $\|AU\|_2 = \|A\|_2$. 

3
(ii) Since $\|A\|_F = (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)^{\frac{1}{2}}$, again from (*), we have

$$\|AU\|_F = (\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2)^{\frac{1}{2}}.$$ 

Therefore, we showed $\|A\|_F = \|AU\|_F$.

7.22
Let $U\Sigma V^T$ be the SVD of $A$. Then prove that $\|U^T AV\|_F^2 = \sum_{i=1}^p \sigma_i^2$, where $\sigma_i$ are the singular values of $A$.

-Solution-
From the Singular Value Decomposition, for an $m \times n$ matrix $A$, there exists $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that $A = U\Sigma V^T$ where $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_p) \in \mathbb{R}^{m \times n}$ and $p = \min(m, n)$. Since $U$ and $V$ are orthogonal matrices, we have

$$A = U\Sigma V^T \Rightarrow U^T AV = \Sigma.$$

Now taking the Frobenius norm on $U^T AV = \Sigma$, we have

$$\|U^T AV\|_F^2 = \|\Sigma\|_F^2 \quad \cdots \quad (1)$$

By the definition of the Frobenius norm, $\|A\|_F = \left[\sum_{j=1}^n \sum_{i=1}^m |a_{ij}|^2\right]^{\frac{1}{2}}$, since $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_p)$, we have

$$\|\Sigma\|_F^2 = \sum_{i=1}^p \sigma_i^2 \quad \cdots \quad (2)$$

From (1) and (2), we showed that $\|U^T AV\|_F^2 = \sum_{i=1}^p \sigma_i^2$.

7.23
Let $A$ be an $m \times n$ matrix.
(a) Using the SVD of $A$, prove that
(i) $\|A^T A\|_2 = \|A\|_2^2$;
(ii) $\text{Cond}_2(A^T A) = (\text{Cond}_2(A))^2$;
(iii) $\text{Cond}_2(A) = \text{Cond}_2(U^T AV)$, where $U$ and $V$ are orthogonal.

(b) Let $\text{rank}(A_{m \times n}) = n$, and let $B_{m \times r}$ be a matrix obtained by deleting $(n-r)$ columns from $A$. Then prove that $\text{Cond}_2(B) \leq \text{Cond}_2(A)$.

-Solution-
(a) Using the Singular Value Decomposition, we have an orthogonal matrices $U \in \mathbb{R}^{m \times m}$ and $V \in \mathbb{R}^{n \times n}$ such that $A = U\Sigma V^T$. Without the loss of generality, we assume $m \geq n$.

(i)
By the Singular Value Decomposition, we have 
\[ A = U \Sigma V^T. \]
Therefore, \( \| A \|_2 = \sigma_{\text{max}} = \sigma_1. \) Also, we have 
\[ A^T = V \Sigma^T U^T. \]
Considering \( A^T A \), we have
\[ A^T A = U \Sigma^T V V^T \Sigma U = U \Sigma \Sigma U^T. \]
Since \( \Sigma^T \Sigma = \text{diag}(\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2) \), 
\[ \| A^T A \|_2 = \sigma_{\text{max}}^2 = \sigma_1^2. \]
Therefore, \( \| A \|_2^2 = \| A^T A \|_2. \)

(ii)

\[ \text{Cond}_2(A^T A) = \frac{\sigma_{\text{max}} (A^T A)}{\sigma_{\text{min}} (A^T A)} = \frac{\sigma_{\text{max}} (A^2)}{\sigma_{\text{min}} (A^2)} = \left( \frac{\sigma_{\text{max}} (A)}{\sigma_{\text{min}} (A)} \right)^2 = \text{Cond}_2(A)^2. \]

(iii)

By the Singular Value Decomposition, we have
\[ A = U \Sigma V^T \Rightarrow U^T AV = \Sigma \]
Since \( \text{Cond}_2(\Sigma) = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \), we have
\[ \text{Cond}_2(U^T AV) = \text{Cond}_2(\Sigma) = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}}. \quad \cdots \quad (1) \]
Also, \( \text{Cond}_2(A) = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} \). Therefore, we have \( \text{Cond}_2(A) = \text{Cond}_2(U^T AV) \).

(b)

Let
\[ A_{m \times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}_{m \times n} \]
Then we have
\[ B_{m \times r} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{pmatrix}_{m \times r} \]
Since \( \text{rank}(A) = n \), the number of nonzero singular values of \( A \) is \( n \). This tells us that \( \sigma_{\text{min}} = \sigma_n \leq \sigma_r \). Therefore, we have
\[ \text{Cond}_2(A) = \frac{\sigma_{\text{max}}}{\sigma_{\text{min}}} = \frac{\sigma_1}{\sigma_n} \geq \frac{\sigma_1}{\sigma_r} = \text{Cond}_2(B). \]