1. (a) Using Legendre polynomials of degree 1, 2, and 3, find the least-squares approximation for the function $e^{-x}$ on $[2, 4]$.

(b) Find the quadratic least-squares approximation for $f(x) = \sin \pi(x)$ on $[0, 1]$ using Chebyshev polynomials.
2. (a) Show that the Chebyshev polynomial $T_k(x)$ of degree $k \geq 1$ has $k$ simple zeros in the interval $[-1, 1]$ at

$$x_j = \cos \left( \frac{2j - 1}{2k} \pi \right), \ j = 1, 2, \ldots, k$$

and $T_k(x)$ has extreme points at $z_j = \cos \left( \frac{j\pi}{k} \right), \ j = 0, 1, \ldots, k$.

(b) What are the optimal nodes for interpolating with a polynomial of degree 2 on $[2, 5]$?

(c) Using the zeros of $\tilde{T}_3(x)$, construct an interpolating polynomial of degree 2 for $f(x) = e^x$ on $[2, 5]$. 

2
3. (a) Consider the Chebyshev least-squares approximation of \( f(x) \) on \([-1,1]\) in the form

\[
P_m(x) = \sum_{j=0}^{m} a_j T_j(x)
\]

Find the expressions for \( a_j \) in terms of \( T_j(x) \) and \( f(x) \).

(b) Show that for \( f(x) = \arccos x \) on \([-1,1]\), the expression for \( a_j \) in 3(a) is given by

\[
a_j = \begin{cases} 
-\frac{4}{\pi} j^2 & \text{when } j \text{ is odd} \\
0 & \text{when } j \text{ even}
\end{cases}
\]
4. Using the 4th degree polynomial $p_4(x)$ for $\sin x$ (obtained from a power series expansion), find a polynomial $p_3(x)$ of degree 3 such that

$$\max_{1 \leq x \leq 1} |p_4(x) - p_3(x)|$$

will be minimum.

What is the total error (sum of the truncation error + error by Cheysher approximation)?