MATH 435
Spring 2013
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Review for
Exam # 2
1. Learn the derivations of the composite Trapezoidal and Simpson’s rules and those of their error formulas.

2. Construct a quadratical formula of the form

\[ \int_{-1}^{1} f(x)dx = A_0 f(-1) + A_1 f(0) + A_2 f(1) \]

Which is exact for all polynomials of as high degree as possible.

3. Determine the values of \( h \) and \( N \) to approximate the following integrals to within \( \varepsilon = 10^{-8} \) using both the composite trapezoidal and Simpson’s rules:

   (a) \( \int_0^2 \frac{1}{x + 4} \, dx \)

   (b) \( \int_0^1 e^{-x^2} \, dx \)

   (c) \( \int_0^2 e^x \sin 3x \, dx \)

   (d) \( \int_0^\pi \sin x \, dx \)
4. Let

\[ f(x) = x, \quad 0 \leq x \leq \frac{1}{2} \]

\[ = 1 - x, \quad \frac{1}{2} \leq x \leq 1. \]

Calculate approximations of \( \int_0^1 f(x) \, dx \) using

(a) The trapezoidal rule over the interval \([0, 1]\)

(b) The trapezoidal rule over the interval \([0, \frac{1}{2}]\) and \([\frac{1}{2}, 1]\)

(c) Simpson's rule over \([0, 1]\)

Compare the results
5. Learn the derivation of Gaussian Quadrature formula with \( n = 2 \)

6. Determine the constants \( \alpha, \beta, \nu \) and \( \delta \) so that the quadrature formula

\[
\int_{-1}^{1} f(x)dx = \alpha f(-1) + \beta f(1) + \nu f'(-1) + \delta f'(1)
\]

has a degree of precision 3.

7. Approximate the following integrals using Gaussian Quadrature with \( n = 2 \)

\[
\text{(a)} \quad \int_{0}^{\frac{\pi}{4}} e^{3x} \sin 2x \, dx
\]

\[
\text{(b)} \quad \int_{3}^{3.5} \frac{x}{\sqrt{x-9}} \, dx.
\]
8. Learn all the properties of Chebyshev polynomials and their derivations.

9. Suppose that \( P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \) needs to be approximate by a polynomial \( P_{n-1}(x) \) of degrees \( n - 1 \) such that

\[
\max_{-1 \leq x \leq 1} |P_n(x) - P_{n-1}(x)|
\]

is as small as possible. Then prove that

\[
P_{n-1}(x) = P_n(x) - a_n \tilde{T}_n(x),
\]

where \( \tilde{T}(x) \) is the monic Chebyshev polynomial of degree \( n \).

What is the minimum value?

Apply the above result to the polynomial \( P_4(x) = 1 + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{24} \) in \([-1, 1]\)

Find the minimum value.

10. Find the best possible choices of nodes for interpolation of \( f(x) = e^x \) with a polynomial of degree at most 3 in \([2, 3]\)
11. Learn the Normal Equations Method for least-square approximations of a set of discrete data and of a function in an arbitrary interval.

12. Construct an interpolating polynomial of degree at most three to interpolate \( f(x) = e^{-x^2} \) using zeros of an appropriate Chebyshev polynomial on \([2,3]\). Repeat the problem with other functions, \( f(x) = \sin 3x \), and \( f(x) = \frac{1}{x} \).

13. Learn the process of power series economization with the zeros of Chebyshev polynomials. Practice the process with \( f(x) = \sin x \) starting with a power series expansion of order 6 on \([-1,1]\) and then reducing the degree while keeping errors less than 0.01.

14. Learn the derivations of

   (a) Taylor's method of order \( k \)

   (b) Heun's method

   (c) Midpoint method

   (d) Modified Euler's method
15. Use Euler's method to approximate the solutions of

(a) \( y' = y = t^2 + 1, \, 0 \leq t \leq 2, \, y(0) = 0.5, \) with \( h = 0.5 \)

(b) \( y' = \frac{y}{t} - \left( \frac{y}{t} \right)^2, \, 1 \leq t \leq 2, \, y(1) = 1 \) with \( h = 0.1 \).

Compute both the local and global error bounds in each case for Euler's method.

16. Consider solving the IVP: \( y' = -2y, \, 0 \leq t \leq 1, \, y(0) = 1 \), using Euler's method.

Find an upper bound on the local error at \( t = 1 \) in terms of the step size \( h \). How small does \( h \) have to be to obtain an accuracy of \( \epsilon = 10^{-5} \) at \( t = 1 \)?

17. Apply Midpoint method, the modified Euler's method and Heun's method to

\[ y' = -y + t + 1, \, 0 \leq t \leq 1, \, y(0) = 1, \, h = 0.01 \]

to compute approximations of \( y(0.01), y(0.02), \ldots, y(1) \).

Tabulate the results of approximations by each method for \( t_i = 0, 0.1, 0.2, \ldots, 1 \)

Compare the errors of those approximations with the actual values.
18. Derive Euler's Trapezoidal predictor-corrector method. Practice this method with

\[ y' = -ty^2, \quad y(2) = 1 \]

\[ h = 0.1 \text{ on } [2, 3] \]

Does the inner iteration converge for this value of \( h \)? Give reasons for your answer.