MATH 435

Spring 2009

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Review for

Exam # 2
1. Learn the derivations of the composite Trapezoidal and Simpson’s rules and those of their error formulas.

2. Derive Simpson’s rule by using

$$\int_{x_0}^{x_2} f(x)dx = a_0f_0 + a_1f_1 + a_2f_2 + K f^{(4)}(n).$$

(Hint: Find $a_0$, $a_1$, and $a_2$ so that Simpson’s rule is exact for all polynomials of degree less than or equal to 3 and then find $K$ by applying the integration formula with $f(x) = x^4$).

3. Determine the values of $h$ and $N$ to approximate the following integrals to within $\epsilon = 10^{-5}$ using both the composite trapezoidal and Simpson’s rules:

(a) \( \int_0^2 \frac{1}{x + 4} \, dx \)

(b) \( \int_0^1 e^{-x^2} \, dx \)

(c) \( \int_0^2 e^{Rt} \sin 3x \, dx \)

(d) \( \int_0^\pi \sin x \, dx \)
4. Apply Romberg integration to approximate \( \int_0^1 x^4 dx \) until \( R_{n-1,n-1} \) and \( R_{nn} \) agree to within \( 10^{-4} \).

5. Romberg integration is used to approximate

\[
\int_0^1 \frac{x^2}{1 + x^3} dx.
\]

If \( R_{11} = 0.25 \), \( R_{22} = 0.2315 \), find \( R_{21} \).

6. Let

\[
f(x) = \begin{cases} 
    x & 0 \leq x \leq \frac{1}{2} \\
    1 - x & \frac{1}{2} \leq x \leq 1.
\end{cases}
\]

Calculate approximations of \( \int_0^1 f(x) dx \) using

(a) The trapezoidal rule over the interval \([0, 1]\)

(b) The trapezoidal rule over the interval \([0, \frac{1}{2}]\) and \([\frac{1}{2}, 1]\).

(c) Simpson’s rule over \([0, 1]\)

7. Learn the derivation of Gaussian Quadrature formula with \( n = 2 \).
8. Determine the constants $\alpha, \beta, \nu$ and $\delta$ so that the quadrature formula

$$\int_{-1}^{1} f(x)dx = \alpha f(-1) + \beta f(1) + \nu f'(1) - \delta f'(1)$$

has degree of precision 3.

9. Approximate the following integrals using Gussian Quadrature with $n = 2$

(a) $\int_{0}^{\pi} 4 e^{3x} \sin 2xdx$

(b) $\int_{3}^{3.5} \frac{x}{\sqrt{x-9}} dx$

10. Learn all the properties of Legendre and Chebyshev polynomials.

11. Suppose that $P_n(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ needs to be approximated by a polynomial $P_{n-1}(x)$ of degree $n - 1$ such that

$$\max_{-1 \leq x \leq 1} |P_n(x) - P_{n-1}(x)|$$

is as small as possible. Then prove that

$$P_{n-1}(x) = P_n(x) - a_n \tilde{T}_n(x).$$

What is the minimum value?

Apply the above result to polynomial $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{4} + \frac{x^4}{24}$ in $[-1, 1]$.

Find the minimum value.

12. What are the best possible choices of nodes for interpolation of $f(x) = e^x$ with a polynomial of degree at most 3 in $[2, 3]$?

14. Construct an interpolating polynomial of degree at most three to interpolate \( f(x) = e^{-x^2} \) using zeros of an appropriate Chebyshev polynomial on \([2, 3]\). Repeat the problem with other functions, \( f(x) = \sin 3x \), and \( f(x) = \frac{1}{x} \).

15. Learn the process of power series economization with the zeros of Chebyshev polynomials. Practice the process with \( f(x) = \sin x \) starting with a power series expansion of order 6 on \([-1, 1]\) and then reducing the degree while keeping error less than 0.01.

16. Learn the derivation of
   
   (a) Taylor’s method of order \( n \)

   (b) Runge-Kutta method of order 2

   (c) Heun’s method
17. Use Euler’s method to approximate the solutions of

(a) \( y' = y - t^2 + 1, \ 0 \leq t \leq 2, \ y(0) = 0.5, \) with \( h = 0.5 \)

(b) \( y' = \frac{y}{t} - \left( \frac{y}{t} \right)^2, \ 1 \leq t \leq 2, \ y(1) = 1 \) with \( h = 0.1. \)

Compute both the local and global error bounds in each case.

18. Given the IVP: \( y' = -2y, \ 0 \leq t \leq 1, \ y(0) = 1. \)

Find an upper bound on the error at \( t = 1 \) in terms of the step size \( h. \) How small does \( h \) have to be to obtain an accuracy of \( e = 10^{-5} \) at \( t = 1? \)

19. Show that the Midpoint method, the modified Euler’s method and Heun’s method
applied to

\( y' = -y + t + 1, \ 0 \leq t \leq 1, \ y(0) = 1 \)

give the same approximations for any choice of \( h. \)