Compatible Orderings on the Bicyclic Semigroup*

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Abstract

In this paper we determine those compatible partial orders on the bicyclic semigroup $B$ which turn it into a semilatticed semigroup. We shall see that there are exactly four distinct compatible total orderings on $B$. There are the only compatible orderings which turn $B$ into a lattice ordered semigroup.

On a group every compatible semilattice ordering is a lattice ordering. However this is not the case with inverse semigroups. Indeed, the situation regarding semilattice orderings on the bicyclic semigroups is much richer. There are four infinite families of compatible semilattice orderings on $B$. Two of these families turn $B$ into a $\lor$-semilatticed semigroup; two of the families turn it into a $\land$-semilatticed semigroup.

1. Introduction

A partial order on a semigroup $S$ is said to be compatible with multiplication, and $S$ is said to be a partially ordered semigroup if, for all $a, b, c \in S$,

$$a \leq b \text{ implies } ca \leq cb \text{ and } ac \leq bc.$$ 

The semigroup $S$ is said to be a $\lor$-semilatticed semigroup, under $\leq$, if $S$ is a $\lor$-semilattice (that is, each pair of elements $a, b \in S$ has a least upper bound $a \lor b \in S$) and, for all $a, b, c \in S$,

$$c(a \lor b) = ca \lor cb, \text{ and } (a \lor b)c = ac \lor bc.$$ 

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There is a well developed theory of partially ordered groups. The following
lemma gives some standard results from the theory of partially ordered groups
which will be relevant to our investigations. These may be found, for example,
in [4].

**Lemma 1.1.** Let $G$ be a partially ordered group. Then, for $a, b \in G$,

(i) $a \leq b$ if and only if $b^{-1} \leq a^{-1}$;

(ii) $a \lor b$ exists if and only if the greatest lower bound $a \land b$ of $a, b$ exists.
Further, $a \lor b$ exists if and only if $a^{-1} \land b^{-1}$ exists. In this case

\[
a \lor b = (a^{-1} \land b^{-1})^{-1} = a(a \land b)^{-1}b
\]

\[
a \land b = (a^{-1} \lor b^{-1})^{-1} = a(a \lor b)^{-1}b.
\]

(iii) $G$ is $\lor$-semilatticed [$\land$-semilatticed] if and only if it is a latticed group
so that, for $a, b, c \in G$

\[
c(a \lor b) = ca \lor cb \text{ and } (a \lor b)c = ac \lor bc
\]

\[
c(a \land b) = ca \land cb \text{ and } (a \land b)c = ac \land bc.
\]

Further, in this case, $G$ is a distributive lattice under $\leq$.

The partial order on a partially ordered group $G$ is determined by the set
$G^+$ of positive elements; that is the set of elements $g \in G$ with $g \geq 1$. The
terminology ‘positive’ is used because it is usual to adopt additive notation for
the binary operation on $G$ so that the identity is denoted by $0$. This means
that $G^+ = \{g \in G : g \geq 0\}$. Then, for $g, h \in G$,

\[g \leq h \text{ if and only if } h - g \in G^+ \text{ or, equivalently, } -g + h \in G^+.
\]

There is a close relationship between compatible partial orders on inverse semi-
groups and compatible partial orders on groups. For suppose that $S$ is a par-
tially ordered inverse semigroup. Then we can define a compatible partial
order on the maximum group homomorphic image $S/\sigma$ of $S$ as follows:

\[a\sigma^i \leq b\sigma^j \text{ if and only if } ea \leq eb \text{ for some } e^2 = e \in S.
\]

If $S$ is $\lor$-semilatticed [$\land$-semilatticed] then it is easy to see that $\sigma$ is a $\lor$
congruence as well. Thus $S/\sigma$ is a lattice ordered group. It follows that the $\sigma$-
class which contains the idempotents is a $\lor$-subsemilattice [$\land$-subsemilattice]
of $S$. Thus, if $S$ is $E$-unitary, the idempotents form a $\lor$-subsemilattice [$\land$
subsemilattice] of $S$.

However there is an essential difference between compatible partial orders
on groups and compatible partial orders on inverse semigroups. This difference
arises from the fact that inversion is an order anti-isomorphism on a partially ordered group:
\[ g \leq h \text{ if and only if } h^{-1} \leq g^{-1}. \]
This is not the case, in general, for inverse semigroups. Indeed, if the analog holds in an inverse semigroup then the idempotents must be trivially ordered.

To make this difference more precise, let \( \leq \) be a compatible partial order on an inverse semigroup \( S \). Then we can define three related compatible partial orders on \( S \) as follows:

\[
\begin{align*}
    a \leq b & \text{ if and only if } b \leq a; \text{ we denote this partial order by } \leq_o; \\
    a \leq b & \text{ if and only if } a^{-1} \leq b^{-1}; \text{ we denote this partial order by } \leq_i; \\
    a \leq b & \text{ if and only if } b^{-1} \leq a^{-1}; \text{ we denote this by } \leq_{io}.
\end{align*}
\]

These prescriptions define an action of the Klein four group \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) on the set of compatible partial orders on \( S \).

**Proposition 1.2.** Let \( \leq \) be a non-trivial compatible partial order on an inverse semigroup \( S \). Then the orbit of \( \leq \) has order two or four. If \( S \) is a group then the orbit of \( \leq \) has order two. If \( S/\sigma \) is not trivially ordered and the set of idempotents of \( S \) is not trivially ordered under \( \leq \) then the orbit of \( \leq \) has order four.

**Proof.** The orbit of \( \leq \) has order one of 1, 2, 4. On the other hand, since \( S \) is not trivially ordered, \( \leq_o \) is different from \( \leq \). Hence the order is two or four.

Suppose that \( S \) is a group. Then \( \leq = \leq_{io} \) so the order is not four; thus it is two. Now suppose that \( S \) is not a group but that \( S/\sigma \) is not trivially ordered and that the set of idempotents is not trivially ordered.

Since \( S/\sigma \) is not trivially ordered, there exist \( x, y \in S \) such that \( x < y \) and \( x\sigma^i < y\sigma^i \). Then \( x^{-1} \leq y^{-1} \) implies \( x\sigma^i \leq y\sigma^i \) which contradicts the fact that \( x\sigma^i < y\sigma^i \) implies \( y\sigma^i < x\sigma^i \). Hence \( \leq \neq \leq_i \). Further, since \( e^{-1} = e \) for each idempotent \( e \in S \) and the idempotents are not trivially ordered, \( \leq \neq \leq_{io} \). But then, since \( \leq \neq \leq_o, \leq_i \neq \leq_{oi} = \leq_{io} \) so that the orbit has order at least three. Thus it has order four.□

**Corollary 1.3.** Let \( S \) be a non-trivial \( \lor \)-semilatticed or a \( \land \)-semilatticed inverse semigroup which is not a group, under a partial order \( \leq \). If \( S/\sigma \) is a non-trivial group then the orbit of \( \leq \) has order four.

**Proof.** As remarked earlier, \( \sigma \) is a \( \lor \)-semilattice congruence so that \( S/\sigma \) is a \( \lor \)-semilattice ordered group and so has non-trivial partial order. Now let \( e, f \) be distinct idempotents of \( S \) — these exist since \( S \) is not a group; let \( a = e \lor f \). Then \( e, f \leq a \) implies \( e, f \leq a^2 \). But \( a^2 = e \lor ef \lor f = a^3 \) so that \( a^2 \) is an idempotent different from \( e, f \). Thus the idempotents of \( S \) are not trivially ordered. Hence, by the proposition, the orbit of \( \leq \) has order four.□

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Remark 1. If $S$ is a $\vee$-semilatticed inverse semigroup then $S/\sigma$ is a $\vee$-semilatticed group; thus it is a lattice ordered group. However, it does not follow that $S$ is a lattice ordered semigroup. Indeed, as we shall see, it is possible for $S$ to be a lattice, even a distributive lattice, under $\leq$ without being a lattice ordered semigroup.

Remark 2. Suppose that $S$ is a $\vee$-semilatticed $[\wedge$-semilatticed] inverse semigroup under a partial order $\leq$. Then $S$ is a $\vee$-semilatticed semigroup under $\leq_i$; indeed

$$a \vee_i b = (a^{-1} \vee b^{-1})^{-1}.$$  

On the other hand, $S$ is a $\wedge$-semilatticed semigroup under $\leq_o$ with $a \wedge_o b = a \vee b$ while $S$ is also a $\wedge$-semilatticed semigroup under $\leq_{io}$ with

$$a \wedge_{io} b = (a^{-1} \vee b^{-1})^{-1}.$$  

Following McFadden [7], we shall define a partially ordered semigroup $S$ to be a Dubreil-Jacotin semigroup if there is an isotone homomorphism $\theta$ of $S$ onto a partially ordered group $G$, with identity $1$, such that the set $\{x \in S : x\theta \leq 1\}$ has a greatest element $\xi$. In this case, Blyth [1] has shown that $\xi$ is unique and that $x \leq \xi$ if and only if there exists $y \in S$ such that $xy \leq y$ or $yx \leq y$. If, in addition, $S$ is a regular semigroup then it is easy to see that $\xi$ is an idempotent. If it is the identity of $S$ then $S$ is said to be an integrally closed Dubreil-Jacotin semigroup. McFadden [7] has shown that integrally closed regular Dubreil-Jacotin semigroups are necessarily E-unitary inverse monoids and has used the structure theorem for such semigroups to determine the order structure of such semigroups under the hypothesis that

$$x \leq y \text{ implies } xx^{-1} \leq yy^{-1}.$$  

A partially ordered inverse semigroup $S$ in which this implication holds is said to be right amenable ordered; left amenable ordered inverse semigroups are defined in a dual fashion. Amenability provides a means by which the order structure of a partially ordered semigroup can be related to the algebraic structure of the semigroup. For this reason many results on the structure of partially ordered semigroups, in addition to McFadden’s [7], assume amenability. This is the case in [2],[6] where the structure of amenable ordered inverse semigroups is studied in detail.

The aim of this paper is to describe semilatticed orderings on the bicyclic semigroup $B$. This is the semigroup generated by symbols $a, b$ subject to the relation $ab = 1$ where $1$ is the identity. Then the elements of $B$ can be uniquely expressed in the form $b^r a^s$ under the multiplication

$$b^r a^s b^u a^v = b^{r+u} a^{s+v}.$$  

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where 
\[ p = (u \lor s) - s + r \quad \text{and} \quad q = (s \lor u) - u + v, \]
and \( u \lor s = \text{max}(u, s) \) is the join in the group \( \mathbb{Z} \) of integers under the usual ordering. We shall see, in Section 6, that although the bicyclic semigroup admits an infinite number of compatible semilattice orderings only two of these are right amenable, and only two are left amenable. It follows from [6] that \( B \) admits no compatible semilattice ordering which is both left and right amenable.

**Proposition 1.4.** Let \( B = \{ b^r a^s : r, s \geq 0 \} \) be the bicyclic semigroup where \( ab = 1 \). Suppose that \( B \) is a \( \lor \)-semilatticed \( [\land \)-semilatticed] semigroup under some compatible partial order \( \preceq \). Then one of the following four (mutually exclusive) conditions must hold:

(i) \( a > 1 \) and \( ba > 1 \);

(ii) \( a > 1 \) and \( ba < 1 \);

(iii) \( a < 1 \) and \( ba > 1 \);

(iv) \( a < 1 \) and \( ba < 1 \).

**Proof.** Let us assume that \( B \) is a \( \lor \)-semilatticed semigroup; the other case follows analogously. Then

\[ a(a \lor 1) = a^2 \lor a = (a \lor 1)a, \]

so that \( a \lor 1 \) commutes with \( a \). Because of the form of multiplication in \( B \) this implies that \( a \lor 1 = a^m \) for some non-negative integer \( m \). Likewise, \( b \lor 1 = b^n \) for some non-negative integer \( n \). But then

\[ ab^n = a(b \lor 1) = ab \lor a = 1 \lor a = a^m, \]

which implies \( n = 0, m = 1 \) or \( n > 0 \) and \( b^{n-1} = a^m \); so that \( n - 1 = m = 0 \). That is, \( n = 1, m = 0 \). Hence either \( a \lor 1 = a \) or \( a \lor 1 = 1 \); equivalently, \( a > 1 \) or \( a < 1 \).

Now consider \( ba \lor 1 \). We have \( a(ba \lor 1) = aba \lor a = a \) so that, since \( B \) is E-unitary, \( ba \lor 1 = e \) is idempotent with \( ae = a \). This implies \( e = 1 \) or \( e = ba \). Thus either \( ba < 1 \) or \( ba > 1 \).

The proof of Proposition 1.4 shows that, under the conditions of the proposition, \( a > 1 \) implies \( 1 \lor b = (1 \lor a)b = ab = 1 \) so that \( b < 1 \), and dually \( a < 1 \) implies \( b > 1 \) and vice versa. Although we know from Proposition 1.2 that \( x < y \) does not imply \( y^{-1} < x^{-1} \) in \( B \), \( a < 1 \) under any compatible partial order on \( B \) if and only if \( 1 < b \).

**Lemma 1.5.** Let \( \preceq \) be a compatible partial order on the bicyclic semigroup \( B \). Then

(i) \( a > 1 \) if and only if \( b < 1 \);
(ii) \( b > 1 \) if and only if \( a < 1 \).

**Proof.** Suppose \( a > 1 \). Then \( 1 = ab \geq 1.b = b \) so that, since \( b \neq 1, b < 1 \). Similarly \( b < 1 \) implies \( 1 = ab \leq a.1 = a \) so that \( a > 1 \). Thus (i) holds. Dually, so does (ii). □

Any partial order on \( B \) which satisfies one of the four conditions of Proposition 1.4 belongs to the orbit of a partial order \( \leq \) in which \( a > 1, ba > 1 \). Thus, if we wish to describe partial orders under which \( B \) is a \( \lor \)-semilatticed semigroup or a \( \land \)-semilatticed semigroup, it suffices to describe partial orders with \( a > 1, ba > 1 \). To end this section, we shall apply this insight to prove that \( B \) admits precisely four different compatible total orders. By Proposition 1.4, this is equivalent to showing that there is exactly one such partial order with \( a > 1, ba > 1 \).

**Proposition 1.6.** \( B \) admits exactly one compatible total order \( \leq \) with \( a > 1, ba > a \). It is defined by

\[ b^sa^s \geq b^ua^v \text{ if and only if } s - r > v - u \text{ or } s - r = v - u \text{ and } r \geq u. \]

**Proof.** Direct calculation shows easily that the partial order described in the statement of the theorem turns \( B \) into a totally ordered inverse semigroup in which \( a > 1, ba > 1 \). On the other hand, Torù Saitô [8] has shown that if \( S \) is a totally ordered \( E \)-unitary inverse semigroup then, for \( x, y \in S \),

\[ x \geq y \text{ if and only if } x\sigma^1 > y\sigma^1 \text{ or } x\sigma^1 = y\sigma^1 \text{ and } xx^{-1} \geq yy^{-1}. \]

If \( a > 1 \) then, as ordered groups, \( B/\sigma \approx \mathbb{Z} \) under the usual ordering, with \( b^sa^s \sigma^1 = s - r \). Also, since \( ba > 1 \), it is easy to see that \( b^m a^m \geq b^n a^n \) if and only if \( m \geq n \). Hence, since \( (b^sa^s)^{-1} = b^sa^r \), it follows from Saitô's characterization that

\[ b^sa^s \geq b^ua^v \text{ if and only if } s - r > v - u \text{ or } s - r = v - u \text{ and } r \geq u. \] □

**Theorem 1.7.** The bicyclic semigroup \( B \) admits precisely four compatible total orders; these are defined by the following four relations:

(i) \( b^sa^s \geq b^ua^v \) if and only if \( s - r > v - u \) or \( s - r = v - u \) and \( r \geq u \);

(ii) \( b^sa^s \geq b^ua^v \) if and only if \( s - r > v - u \) or \( s - r = v - u \) and \( r \leq u \);

(iii) \( b^sa^s \geq b^ua^v \) if and only if \( s - r < v - u \) or \( s - r = v - u \) and \( r \geq u \);

(iv) \( b^sa^s \geq b^ua^v \) if and only if \( s - r < v - u \) or \( s - r = v - u \) and \( r \leq u \).

The results of Theorem 1.7 also follow directly from the analysis of compatible semilatticed orderings on \( B \) which is contained in Sections 2,3.
2. Cones in the Bicyclic Semigroup

Suppose that \( \leq \) is a compatible partial ordering on the bicyclic semigroup \( B \) in which \( a > 1, ba > 1 \). The main result of this section is that \( \leq \) is determined by its negative cone \( K = \{ x \in B : x \leq 1 \} \). The first lemma shows that the positive cone \( \{ x \in B : x \geq 1 \} \) is the same for all such partial orders.

**Lemma 2.1.** Let \( \leq \) be a compatible partial order on \( B \) in which \( a > 1 \). Then

(i) \( a^m \leq a^n \) if and only if \( m \leq n; b^m \leq b^n \) if and only if \( m \geq n; \)

(ii) \( b^r a^s \leq b^v a^u \) implies \( s - r \leq v - u; \)

(iii) if, in addition, \( ba > 1 \), then \( b^r a^s \geq 1 \) if and only if \( s \geq r. \)

**Proof.** (i). Since \( a > 1 \), it is easy to see that \( m \leq n \) implies \( a^m \leq a^n \). For the converse, suppose that \( m > n \). Then \( a^{m-n} > 1 \) and so

\[
a^m = a^{(m-n)+n} = a^{m-n}a^n \geq a^n.
\]

Since \( m \neq n \) implies \( a^m \neq a^n \) it follows that \( a^m > a^n \). Thus \( a^m \leq a^n \) implies \( m \leq n \). The result concerning \( b \) is obtained dually since \( b < 1 \).

(ii). Suppose that \( b^r a^s \leq b^v a^u \). Then, multiplying on the left by \( a^{r+u} \) we get

\[
a^{r+u-r+s} \leq a^{r+u-u+v}.
\]

Thus, from (i), \( r \vee u - r + s \leq r \vee u - u + v; \) that is \( s - r \leq v - u. \)

(iii). Suppose that \( b^r a^s \geq 1 \). Since \( a > 1 \), it follows from (ii) that \( s \geq r \). Conversely, if \( s \geq r \) then \( a^s \geq a^r \) which implies \( b^r a^s \geq b^r a^r \). But \( ba > 1 \) so that \( b^r a^r \geq 1 \) for all non-negative integers \( r \). Thus \( b^r a^s \geq 1. \)

The result of (iii) is a direct verification of the fact, alluded to above, that the canonical homomorphism \( \sigma^1 : B \to B/\sigma \approx \mathbb{Z} \) is order preserving. For, since \( a > 1 \), the ordering on \( B/\sigma \approx \mathbb{Z} \) is the usual ordering.

It is helpful to use a graphical representation when considering compatible partial orders on \( B \). Thus the element \( b^r a^s \) is represented by the point \( (s, r) \) in the first quadrant of the plane. An arrow is drawn from \( (s, r) \) to \( (u, v) \) whenever \( b^r a^s \leq b^u a^v \) to obtain the graph of the partial order. Given a set of inequalities, the graph of the compatible partial order is obtained by closing the corresponding graph by repeatedly drawing the arrows which result by left and right multiplication of the inequalities by \( a, b \). Then \( b^r a^v \leq b^u a^v \) if and only if there is a directed path in the graph from \( (s, r) \) to \( (v, u) \).

**Example 1.** Consider the compatible partial order on \( B \) generated by the inequalities \( a > 1, ba > 1 \). Then the graph contains right facing horizontal arrows from \( (s, r) \) to \( (s + 1, r) \) obtained by pre-multiplying the inequality \( 1 < \)
a by $b^r$ and post-multiplying by $a^s$. It contains vertical down arrows from $(s, r + 1)$ to $(s, r)$ obtained from $1 < a$ by pre multiplying by $b^r$ and post-multiplying by $ba^s$. It also contains diagonal arrows from $(s, r)$ to $(s + 1, r + 1)$ obtained by pre-multiplying the inequality $1 < ba$ by $b^r$ and post-multiplying by $a^s$. Since the resulting graph is closed under multiplication by $b, a$ it is the graph of the compatible partial order generated by the inequalities. Thus we find $b^r a^s \leq b^r a^x$ if and only if there exist non-negative integers $m, n, t$ such that $r + m - n = y$ and $s + m + t = x$. These imply $s - r + t + n = x - y$ so that $s - r \leq x - y$ and $s \leq x$. Conversely, if these inequalities hold, we obtain a solution to the equations by putting

$$m = x - s, n = r + m - y = (x - y) - (s - r) \geq 0, t = 0.$$  

Hence

$$b^r a^s \leq b^r a^x$$

if and only if $y \leq x + (r - s)$ and $x \geq s$.

In this case the negative cone is $K = \{ b^r : r \geq 0 \}$. Although this graphical procedure meshes well with partial orders on $B$ this is not directly the case with multiplication. For example, from drawing the picture, it is easy to see that $B$ is a lattice under $\leq$. However $B$ is not a semilatticed semigroup under this partial ordering. For, graphically, one can see that $b^3 a \land b = b^2$ but

$$1 = a^2 (b^3 a \lor b) \neq a^2. b^3 a \land a^2.b = ba \land a = ba,$$

so that $B$ is not a $\land$-semilatticed semigroup. This means that we shall have to use algebraic manipulations, in addition to graphical intuition, in analyzing partial orders on $B$.

**Lemma 2.2.** Let $\leq$ be a compatible partial order on the bicyclic semigroup $B$ with $1 < a, ba,$ and let $K = \{ x \in B : x \leq 1 \}$. Then $b^r a^s \leq b^v a^w$ if and only if $s - r \leq v - u$ and either $s \leq v$ or $s > v$ and $b^{-u} a^{s-w} \in K$.

**Proof.** Suppose that $b^r a^s \leq b^u a^v$. Then, from Lemma 2.1, $s - r \leq v - u$. Suppose that $s > v$. Then $s - v \leq r - u$ implies $r > u$. It follows that

$$b^{-u} a^{s-w} = a^u . b^r a^s . b^v \leq a^u . b^u a^v . b^v = 1$$

so that $b^{-u} a^{s-w} \in K$.

Conversely, suppose that $s - r \leq v - u$ and either $s \leq v$ and $s > v$ and $b^{-u} a^{s-w} \in K$. If $s \leq v$ then either $u < r$ or $u \geq r$. The first possibility implies $b^{-u} < 1$ and $1 \leq a^{v-s}$. Thus $b^{-u} < a^{v-s}$ which implies

$$b^r a^s = b^u . b^{-u} a^s \leq b^u a^{v-s} a^s = b^u a^v.$$  

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On the other hand, if \( u \geq r \), then \( (v - s) - (u - r) = (v - u) - (s - r) \geq 0 \) so that, by Lemma 2.1, (iii), \( b^{u-r}a^{v-s} \geq 1 \) and then \( b^u a^v = b^r b^{u-r} a^{v-s} a^s \geq b^r 1 a^s = b^r a^s \).

Finally, if \( s > v \) and \( b^{s-u} a^{s-v} \in K \), then \( b^s a^s = b^u b^{s-u} a^{s-v} a^v \leq b^u 1 a^v = b^u a^v \). \( \Box \)

The next lemma gives some properties of \( K = \{ x \in B : x \leq 1 \} \) which we shall use later to characterize compatible partial orders, with \( 1 < a, ba \), in terms of their negative cones.

**Lemma 2.3.** Let \( \leq \) be a compatible partial order on \( B \) in which \( 1 < a, ba \) and let \( K = \{ x \in B : x \leq 1 \} \). Then

(i) \( b \in K \);

(ii) \( E \cap K = \{ 1 \} \) where \( E \) denotes the set of idempotents of \( B \);

(iii) if \( cd \in K \) then \( cKd \subseteq K \).

**Proof.** We have already seen that \( b < 1 \) so we need only show that (ii) and (iii) hold.

(ii) Suppose that \( b^r a^r \in K \). Then, since \( ba > 1 \) implies \( b^n a^n > 1 \) for all \( n > 1 \), it follows that \( r = 0 \) so that \( b^r a^r = 1 \). On the other hand, \( 1 \) certainly belongs to \( K \).

(iii). Suppose that \( cd \in K \), where \( c, d \in B \). Then, for \( k \in K \), \( k \leq 1 \) implies

\[
ckd = c.k.d \leq c.1.d = cd \leq 1.
\]

Thus \( cKd \subseteq K \). \( \Box \)

A subset \( K \) of \( B \) which obeys (i), (ii), (iii) will be called a cone. Because \( 1 \in K \), it follows from (iii) that any cone is a subsemigroup of \( B \); in particular, \( b^n \in K \) for any \( n \geq 1 \).

Theorem 2.4 provides a converse to the results in Lemmas 2.2, 2.3. Because of the complexity of the proof, the details are given in a sequence of lemmas.

**Theorem 2.4.** Let \( K \) be a cone in \( B \) and define \( b^r a^s \leq b^u a^v \) if and only if \( s - r \leq v - u \) and either \( s \leq v \) or \( s > v \) and \( b^{s-u} a^{s-v} \in K \). Then \( \leq \) is a compatible partial order on \( B \) such that \( 1 < a, ba \) and \( K = \{ x \in B : x \leq 1 \} \).

**Lemma 2.5.** \( \leq \) is a partial order on \( B \).

**Proof.** Since \( s - r \leq s - r \) and \( s \leq s \), it is clear that \( \leq \) is reflexive. To show that it is anti-symmetric, note that \( b^r a^s \leq b^u a^v \leq b^r a^s \) implies \( s - r \leq v - u \leq s - r \) so that \( s - r = v - u \), and thus \( s - v = r - u \). If \( s \neq v \) then either \( s < v \) or \( v < s \). Since \( b^r a^s \leq b^u a^v \), the first possibility implies

\[
b^{u-r}a^{u-r} = b^{u-r}a^{u-s} \in K \cap E = \{ 1 \}.
\]
This requires $v - s = u - r = 0$ so that $v = s$; a contradiction. Dually, $s > v$ also leads to a contradiction. Hence $\leq$ is anti-symmetric.

To show transitivity, suppose that $b^p a^s \leq b^p a^v \leq b^p a^q$. Then $s - r \leq v - u \leq q - p$. If $s \leq q$ then, from the definition of $\leq$, we have $b^p a^s \leq b^p a^q$ so we may suppose that $s > q$. We consider several subcases.

Case (i): $v \leq q$. Then $v < s$ and so, from the definition of $\leq$, $b^{q-u} a^{s-v} \in K$. Since $b^p a^v \leq b^p a^q$, $(q - p) - (v - u) \geq 0$ and so $b^{(q-p)-(u-v)} a^{(v-u)} \in K$. But then, since

$$\{(q - p) - (v - u)\} + (r - u) = (q - v) + (r - p),$$

we get

$$b^{-(p-q)} a^{(v-q)} = a^{(q-u)} b^{(v-q)} b^{(r-q)} = a^{(q-v)} b^{(q-p)-(u-v)} a^{(v-u)} b^{-(r-q)}$$

while

$$a^{(s-q)} = a^{(v-q)} b^{(v-q)}.$$ 

Thus

$$b^{(r-q)} a^{(s-q)} = a^{(q-v)} b^{(q-p)-(v-u)} b^{(v-q)} a^{(v-u)} b^{(q-v)},$$

where

$$b^{(q-p)-(u-v)} a^{(v-u)} a^{(s-q)} \in K.$$ 

Hence, since $a^{(q-v)} b^{(v-q)} = 1 \in K$, we have

$$b^{(r-q)} a^{(s-q)} \in a^{(q-v)} K b^{(v-q)} \subseteq K.$$ 

In this case, therefore, $\leq$ is transitive.

Case (ii): $q < v$. Then $b^{u-p} a^{v-q} \in K$. If also $v < s$ then $b^{r-p} a^{s-v} \in K$ and so

$$b^{r-p} a^{s-v} = b^{u-q} b^{p-q} a^{v-q} a^{s-v} \in b^{r-q} K a^{s-v} \subseteq K$$

since $b^{r-u} a^{s-v} \in K$. Thus we may suppose $s \leq v$.

We seek non-negative integers $x, y$ such that

$$b^{r-p} a^{s-q} = a^x b^y b^{u-p} a^{v-q} b^{r-x} b^y.$$ 

Then $b^y b^{u-p} a^{v-q} \in K$ and so, since $a^x b^y = 1 \in K$, $b^{r-p} a^{s-q} \in K$ as required.

Since $v \geq s$ we may take $x = v - s \geq 0$. Set

$$y = (r - u) + x = (r - u) + (v - s) = (v - u) - (s - r) \geq 0$$

since $b^r a^s \leq b^u a^v$. Then the equation

$$a^x b^y b^{u-p} a^{v-q} b^{r-x} b^y = a^{v-s} b^{u+u-p} a^{v-q} b^{v-s} = a^{v-s} b^{(v-s) + (r-p)} a^{(s-q) + (v-s)} b^{v-s}$$

is satisfied so that $\leq$ is transitive in this case also. $\square$
Now that we have shown that $\leq$ is a partial order, the next step is to show that it is compatible. Since $B$ is generated by $a, b$, it suffices to prove that $\leq$ is compatible with multiplication on left and right by $a, b$. The easy cases are left multiplication by $b$ and right multiplication by $a$. We deal with these cases in the next lemma.

**Lemma 2.6.** $\leq$ is compatible with multiplication on the left by $b$ and on the right by $a$.

**Proof.** Suppose that $b^r a^s \leq b^u a^v$. Then $b.b^r a^s = b^{r+1} a^s, b.b^u a^v = b^{u+1} a^v$. Thus

$$s - (r + 1) = (s - r) - 1 \leq (v - u) - 1 = v - (u + 1).$$

If $s \leq v$ then, from the definition of $\leq$, $b^{r+1} a^s \leq b^{u+1} a^v$, so we may suppose that $s > v$. Then $b^{(r+1)-(u+1)} a^{s-v} = b^{r-u} a^{s-v} \in K$ so that $\leq$ is compatible in this case also. The situation involving multiplication on the right by $a$ follows in a dual fashion. □

**Lemma 2.7.** $\leq$ is compatible with respect to left multiplication by $a$.

**Proof.** Suppose that $b^r a^s \leq b^u a^v$, which requires that $s - r \leq v - u$. We consider four cases.

Case $(i)$: $r = u = 0$. Then $s \leq v$ and so $s + 1 \leq v + 1$. Thus

$$a.b^r a^s = a^{s+1} \leq a^{v+1} = a.b^u a^v,$$

which shows that $\leq$ is compatible with left multiplication by $a$ in this case.

Case $(ii)$: $r > 0, u > 0$. Then $s - r \leq v - u$ implies

$$s - (r - 1) = (s - r) + 1 \leq (v - u) + 1 = v - (u - 1).$$

If $s \leq v$ then, by definition,

$$a.b^r a^s = b^{r-1} a^s \leq b^{u-1} a^v = a.b^u a^v.$$

On the other hand, if $s > v$,

$$b^{(r-1)-(u-1)} a^{s-v} = b^{r-u} a^{s-v} \in K$$

so that $a.b^r a^s \leq a.b^u a^v$ in this case too.

Case $(iii)$: $r = 0, u > 0$. Then $a^s \leq b^u a^v$ so that $s \leq v - u < v$. Thus $s + 1 \leq v$ and so, since $v - (u - 1) = v - u + 1 \geq s + 1$,

$$a.a^s = a^{s+1} \leq b^{u-1} a^v = a.b^u a^v.$$
Case (iv): $r > 0, u = 0$. Then $b^r a^s \leq a^v$ which implies $s - r \leq v$. If $s \leq v + 1$ then, since $s - (r - 1) = (s - r) + 1 \leq v + 1$,

$$a.b^r a^s = b^{r - 1} a^s \leq a^{v + 1} = a^v.$$ 

On the other hand, if $s > (v + 1)$, then $v < s$ and so $b^r a^s \in K$. But then

$$b^{r - 1} a^{s - (v + 1)} = a.b^r a^{s - v}.b \in aKb \subseteq K,$$

since $ab = 1 \in K$, so that $a.b^r a^s \leq a.b^r a^s$. Hence $\leq$ is compatible with left multiplication by $a$ in all four cases. The proof that $\leq$ is compatible with right multiplication by $b$ is similar; it is omitted. □

To complete the proof of the theorem we need only prove that $K = \{x \in B : x \leq 1\}$. From the definition of the partial order, $b^r a^s \leq 1 = b^0 a^0$ if and only if $s \leq r$ and either $s \leq 0$, so $s = 0$, or $s > 0$ and $b^r a^s = b^{s - 0} a^{s - 0} \in K$. Since $b \in K$, this means that $b^r a^s \leq 1$ if and only if $b^r a^s \in K$. □

A useful observation concerning cones is the following: if $b^r a^s \in K \setminus \{1\}$ for some cone $K$ then $r - s > 0$; that is, $(b^r a^s)\sigma^1 < 0$. This is because $K$ is the set of elements $\leq 1$ under the corresponding partial order. But $b^r a^s \in K$ implies $r - s \geq 0$ by the isotone property of $\sigma^1$. However $r = s$ means that $b^r a^s$ is an idempotent in $K$ so that, since 1 is the only idempotent in $K$, we get a contradiction. Hence $r > s$.

There is a smallest cone, namely $\{b^n : n \geq 0\}$, and also a largest one $U = \{b^r a^s : r > s\} \cup \{1\}$, as the next proposition shows. Thus the set of cones form a complete lattice under inclusion; indeed, from the form of partial orders in relation to their cones, it is easy to see that the lattice of cones is isomorphic to the lattice of compatible partial orders $\leq$ on $B$ with $1 < a, ba$.

In particular, given any subset $X \subseteq \{b^r a^s : r > s\} \cup \{1\}$, there is a smallest cone $K(X)$ which contains $X$. We say that $K(X)$ is the cone generated by $X$. In the case that $X = \{b^r a^s\}$, we shall write $K(b^r a^s)$ for $K(X)$. Thus $K(1) = \{b^r : n \geq 0\}$.

We shall find it convenient to have the following simple lemma on factorizations, and multiplication, in $B$.

**Lemma 2.8.** Let $b^r a^s \in B$.

(i) If $b^r a^s = b^r a^s.b^u a^v$ then either $r = x$ and $y = s - v + u$ where $s \geq v$, or $s = v$ and $u = r - x + y$ where $r \geq x$;

(ii) $b^r a^s = a^{u+v}b^r a^u b^v a^v$;

(iii) $b^r a^y.b^s a^x.b^t a^p = a^z.b^{r+s+t+q} a^{y+u+v+p} b^r$ where $z = (u+q) \land (y+v) \land (y+q)$.
Proof. (i) We have \( r = u \lor y - y + x, s = y \lor u - u + v \). If \( y \geq u \) then \( x = r \) and \( s = y - u + v \) so that \( y = s - v + u \) where \( s = y \lor u - u + v \geq v \). The case when \( y \leq u \) is similar.

(ii) By definition,
\[
b^r a^y b^u a^v = b^{u \lor y - y + x} a^{v \lor u - u + v} = b^{u \lor y - y + (u + x)} a^{u \lor y - u + y + (v - x)}
\]
\[
= b^{-(u \lor y) + u + x} a^{-(u \lor y) + y + v} = a^{u \lor y} b^{x + u} a^{y + v} b^{u \lor y}.
\]

(iii) This follows from applying (ii) to the repeated product. \( \square \)

Proposition 2.9. \( U = \{ b^r a^s : r > s \} \cup \{1\} \) is the largest cone of \( B \).

Proof. By the remark before the proposition, we need only show that \( U \) has the three defining properties of a cone. In fact, the first two are obvious so suppose that \( b^r a^s, b^r a^q \in U \). Let \( b^r a^s = cd \); we wish to show that \( c b^r a^d, d \in U \). Since \( c \in U \), we can clearly assume that \( b^r a^d \neq 1 \) so that \( p > q \). Lemma 2.8 gives the form of \( c, d \). Suppose for example, \( c = b^r a^{s-v+u}, d = b^r a^v \) where \( s \geq v \). Then
\[
c b^r a^d, d = b^r a^{s-v+u}, b^r a^d, b^r a^v.
\]
The image of this product under \( \sigma^1 \) is
\[
\{(s - v + u) + q + v\} - \{r + p + u\} = \{s + q\} - \{r + p\}
= \{s - r\} + \{q - p\} < 0
\]
since \( r \geq s, p > q \). Thus \( c b^r a^d, d \in U \). The other case is similar. \( \square \)

3. Principal Cones

Later we shall mainly be concerned with cones of the form \( K(b^r a^d) \) where \( c > d \) and it will be important to describe these explicitly. The next lemma gives the form of the cone generated by a cone \( K \) and an element \( b^r a^d \) where \( c > d \). We obtain the cones \( K(b^r a^d) \) by taking \( K = K(1) \).

Proposition 3.1. Let \( K \) be a cone in \( B \). Then \( b^r a^s \) belongs to the cone \( \bar{K} \) generated by \( K \) and \( b^r a^d \), where \( c > d \), if and only if there exist non-negative integers \( m, n \) such that
\[
r + n \geq mc, s + n \geq md, b^{r + n - mc} a^{s + n - md} \in K.
\]

Proof. Suppose that there exist \( m, n \) as in the statement of the proposition. Then
\[
b^r a^s = a^n b^{mc} b^{r + n - mc} a^{s + n - md} a^{md} b^n \in a^n b^{mc} K a^{md} b^n
\]
\[
\subseteq a^n \bar{K} b^n \text{ since } b^r a^d \in \bar{K} \text{ implies } b^{mc} a^{md} \in \bar{K}
\]
\[
\subseteq \bar{K} \text{ since } ab = 1 \in \bar{K}.
\]

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Conversely, let $C$ be the set of all $b^r a^s$ for which there exist such $m, n$. Then, taking $n = 0, m = 1$, we see that $b^r a^d \in C$ and, taking $m = n = 0$, that $K \subseteq C$. Hence, to complete the proof, we need only show that $C$ is a cone.

First, suppose that $b^r a^s \in C$ and let $m, n$ be as in the definition of $C$ so that
\[ b^{r+n-mc} a^{r+n-md} \in K. \]
Then, since $r + n - mc \geq 0$,
\[ r + n - md = (r + n - mc) + m(c - d) \geq m(c - d) \geq 0, \]
since $c > d$, and so, since $b \in K$,
\[ b^{r+n-mc} a^{r+n-md} = b^{r+n-mc} a^{r+n-md} b^m(c-d) \in K. \]
Since $K$ is a cone this implies $r + n - mc = 0$ so that $a^{r+n-md} \in K$ whence $r + n - md = 0$. Hence, since $c > d$, $m = 0$ and thus $r + n = 0$ which implies $r = 0$. Therefore $C \cap E = \{1\}$.

We have seen that $K \subseteq C$ so $b \in C$ and it remains to prove that $pq \in C$ implies $pCq \subseteq C$. In order to do this, it is convenient to note, first, that $aCb \subseteq C$; this is immediate.

Now suppose that $b^r a^s = pq$. Then, by Lemma 2.8, either $p = b^r a^{s-z+w}, s \geq z$ and $q = b^w a^s$, or $p = b^w a^s$ and $q = b^{r-w+z} a^s, r \geq w$.

Consider the former possibility. Then, in the product $p.b^u a^v.q$ we may clearly suppose that $b^u a^v \neq 1$ so that $u > v$. Thus, from Lemma 2.8, since $s \geq z$,
\[ p.b^u a^v.q = b^r a^{s-z+w} b^u a^v b^w a^z = a^x b^{r+u+w} a^{s+w+v} b^z = a^{x-w} b^{r+u} a^{s+v} b^{r-w} \]
where $x = (u + w) \land (s - z + w + v) \land (s - z + w + v) \geq w$. Hence it suffices to prove that $b^u a^v, b^w a^z \in C$ implies $b^{r+u} a^{s+v} \in C$. This, however, follows easily from the definition of $C$.

The proof in the case where $p = b^u a^v$ and $q = b^{r-w+z} a^s, r \geq w$ is analogous. □

**Theorem 3.2.** Let $c, d$ be non-negative integers with $c > d$. Then
\[ K(b^r a^d) = \left\{ b^r a^s : r > s \text{ and } s \leq d \left\lfloor \frac{r-s}{c-d} \right\rfloor \right\} \cup \{1\} \]
where, for any integer $n$, $[n]$ is the greatest integer less than or equal to $n$.

**Proof.** $K(b^r a^d)$ is the cone $C$ generated by adjoining $b^r a^d$ to $K = \{b^p : n \geq 0\}$. Thus, from Proposition 3.1, $b^r a^s \in C$ if and only if there exist non-negative
integers $m, n$ such that $b^{r+n-mc}a^{s+n-md} \in K$. That is, $b^r a^s \in C$ if and only if there exist non-negative integers $m, n$ such that

\[
\begin{align*}
  r + n - mc & \geq 0 \\
  s + n - md & = 0.
\end{align*}
\]

Subtracting these inequalities shows that $r - s \geq m(c - d)$ so that $m \leq \left\lfloor \frac{r - s}{c - d} \right\rfloor$ and

\[
s \leq d \left\lfloor \frac{r - s}{c - d} \right\rfloor.
\]

Hence, since $c > d$, $r \geq s$. But then $r = s$ implies $m = 0$, whence $s + n = 0$, and so $s = 0$. Thus $b^r a^s \in C$ implies $b^r a^s = 1$ or $r > s$ and $s \leq d \left\lfloor \frac{r - s}{c - d} \right\rfloor$.

Conversely, if the inequality in the statement of the theorem holds, we may take

\[
m = \left\lfloor \frac{r - s}{c - d} \right\rfloor, \quad n = md - s
\]

so that the equality $s + n - md = 0$ is valid while

\[
r + n - mc = (r - s) - m(c - d) \geq 0.
\]

Hence $b^r a^s \in C. \square$

4. Co-atoms

Example 2. Proposition 3.1 makes it relatively easy to construct examples of cones and to determine the corresponding compatible partial orders on $B$. For example, let $\lambda$ be any positive integer and set $C = C(\lambda) = \{b^r a^s : s = 0 \text{ or } r \geq s + \lambda\}$. Then it is easy to see that the first two properties of a cone hold. As for the third, we can use the fact that the map $\sigma^i : b^r a^s \to s - r$ is a homomorphism of $B$ onto $\mathbb{Z}$; note that $C$ can be described in terms of $\sigma^i$ as $\{b^r a^s : s = 0 \text{ or } (b^r a^s)\sigma^i \leq -\lambda\}$. Suppose that $cd = b^r a^s, x = b^y a^v \in C$, then

\[
(cx)\sigma^i = (cd)\sigma^i x\sigma^i = (s - r) + (v - u).
\]

From the definition of $C$, either $s = v = 0$, in which case $cx = b^r + u$, or both $s - r, v - u$ are non-positive and at least one is less than or equal to $-\lambda$ so that $(cx)\sigma^i \leq -\lambda$. Hence $cx \in C$.

We have

\[
b^r a^s \leq b^y a^x \text{ if and only if } \begin{cases} 
  \text{either } s \leq x \text{ and } y \leq x + (r - s) \\
  \text{or } x < s, y \leq x + (r - s) + \lambda.
\end{cases}
\]

Thus $b^r a^s \leq a^x$ if and only if $r \leq n$ or $n \geq \lambda$. In particular, $a^\lambda$ exceeds all the idempotents but $b^\lambda a^\lambda \not< a^{\lambda-1}$. Furthermore, if $\lambda > 1$, there is no element $b^\lambda a^x$
with \( b < b^\nu a^x < 1 \). That is, when \( \lambda > 1 \), the partial order has a co-atom in the sense of the following definition.

Suppose that \( \leq \) is a partial order on \( B \). Then we say that an element \( x < 1 \) is a co-atom for \( \leq \) if \( x \) is covered by \( 1 \); that is, \( x < y \leq 1 \) implies \( y = 1 \). The next result shows that every principal cone gives rise to a partial order on \( B \) which has a co-atom.

**Lemma 4.1.** Let \( K = K(b^\nu d) \) where \( c > d > 0 \), and let \( \leq \) be the corresponding compatible partial order on \( B \). Then \( b^\nu a^d \) is a co-atom for \( \leq \).

**Proof.** Suppose that \( x = b^\nu a^s \) is such that \( b^\nu a^d \leq x < 1 \). Then, firstly, \( x \in K \) so that, from Theorem 3.3, \( s \leq d \left\lfloor \frac{r}{\nu} \right\rfloor \). Further, because of the isotone property of \( \sigma^t \), \( d - c \leq s - r \leq 0 \). Thus \( 0 \leq r - s \leq c - d \).

If \( r - s < c - d \) then \( \left\lfloor \frac{r - s}{\nu} \right\rfloor = 0 \); so \( s = 0 \) and \( x = b^\nu \) where, since \( x < 1, r > 0 \). But then the inequality \( b^\nu a^d \leq x \) gives \( b^{r - r} a^d \in K \) and so

\[
d \leq d \left\lfloor \frac{c - r - d}{c - d} \right\rfloor = 0
\]

since \( r > 0 \). This contradicts the fact that \( d > 0 \).

Thus \( r - s = c - d \) and \( s \leq d \), so that

\[
x \geq b^\nu a^d = b^{r + (d - s)} a^{s + (d - s)} = b^{r - d - s} a^{d - s} . a^s \geq b^r . 1 . a^s = x,
\]

since \( b^\nu a^0 \geq 1 \) for all \( n \). Hence \( x = b^\nu a^d \) and \( b^\nu a^d \) is a co-atom, as claimed. \( \square \)

In this paper our main interest is on (principal) partial orders on \( B \) which turn it into a \( \lor \)-semilatticed semigroup. In this case, any co-atom is necessarily unique. Before showing this, we will need the following simple lemma, whose proof depends on the isotonicity of \( \sigma^t \) and is omitted.

**Lemma 4.2.** Let \( \leq \) be a compatible partial order on \( B \) with \( 1 < a, ba \). Suppose that \( 1 \leq x \leq a \); then \( x \) is an idempotent. If \( b \leq y \leq 1 \) then \( y = be \) for some idempotent \( e \in B \).

**Proposition 4.3.** Let \( \leq \) be a compatible partial order on \( B \) in which \( 1 < a, ba \). If there exists a positive integer \( n \) such that \( b^\nu a^n \nless a \) then \( B \) has a co-atom under \( \leq \). Conversely, if \( B \) is \( \lor \)-semilatticed under \( \leq \) and has a co-atom then the co-atom is unique and is equal to \( b^\nu a^r^{-1} \) where \( r = \max\{s \in \mathbb{Z}^+ : b^s a^r < a\} \).

**Proof.** Suppose that \( r = \max\{s \in \mathbb{Z}^+ : b^s a^r < a\} < \infty \). We show first that \( b^\nu a^r^{-1} \) is a co-atom for \( \leq \). Suppose therefore that \( b^\nu a^r^{-1} \leq b^\nu a^v < 1 \). Then, by the isotone property of \( \sigma^t \), \(-1 \leq v - u \leq 0 \). If \( u - v = 0 \) then \( b^\nu a^v \) is an
idempotent strictly less than 1. This contradicts the fact that \( ba > 1 \). Thus \( u = v + 1 \), and on multiplication on the right by \( a \) we get

\[
b^r a^r \leq b^u a^u \leq a.
\]

By the maximality of \( r \), it follows that \( u = r \) and so \( b^r a^{r-1} = b^u a^v \) is a co-atom.

Suppose now that \( b^k a^v \) is a co-atom for a compatible partial order in which \( 1 < a, ba \), and which turns \( B \) into a \( \vee \)-semilatticed semigroup. Then \( u > v \geq 0 \).

Since \( b^r a^v \) is a co-atom, \( b \vee b^r a^v \) is either \( b^r a^v \) or 1. If \( b \vee b^r a^v = b^u a^v \) then, by Lemma 4.2, \( u = v + 1 \). On the other hand, \( b^u a^v \vee b = 1 \) implies

\[
a^u = a^u (b^u a^v \vee b) = a^v \vee a^{u-1}.
\]

But this is impossible since \( a > 1 \) implies \( a^u > a^v, a^{u-1} \). Hence \( u = v + 1 \) so that any co-atom is of the form \( b^{u+1}a \) for some \( u \geq 0 \).

Since \( b^{u+1}a^u < 1 \), we have \( b^r a^u = a, b^{u+1}a^u \leq a \). On the other hand, if \( b^{u+2}a^{u+2} < a \) then multiplication on the right by \( a \) gives \( b^{u+2}a^{u+1} < 1 \) while \( b^{u+1}a^u < b^{u+2}a^{u+1} \). This contradicts the fact that \( b^{u+1}a^u \) is a co-atom. Hence

\[
\max\{s \in \mathbb{Z}^+ : b^s a^s < a \} \leq u + 1 < \infty.
\]

Finally, the uniqueness of co-atoms follows since \( b^{x+1}a^x \leq b^{y+1}a^y \) if and only if \( b^x a^x \leq b^y a^y \), which occurs if and only if \( x \leq y \). So there cannot be more than one co-atom, which, from the first paragraph, must be \( b^r a^{r-1} \) with

\[
r = \max\{s \in \mathbb{Z}^+ : b^s a^s < a \} < \infty.\square
\]

Proposition 4.3 shows that the existence of co-atoms is closely related to the bounding of idempotents from above. The next theorem shows that the same is true for the ascending chain condition on the elements of \( B \) which are less than 1.

**Lemma 4.4.** Let \( \leq \) be a compatible partial order on \( B \) such that \( 1 < a, ba \).

Then \( b^r a^s \leq b^r a^v \) implies \( s - r \leq v - u \). If \( s - r = v - u \) then \( b^r a^s < b^r a^v \) if and only if \( s < v \) or, equivalently, \( r < u \).

**Proof.** That \( b^r a^s \leq b^r a^v \) implies that \( s - r \leq v - u \) is a reiteration of the fact that \( \sigma^r \) is isotone.

Suppose that \( s - r = v - u \). Then \( s < v \) implies

\[
b^u a^v = b^r . b^{u-r} a^{v-s} . a^s \geq b^r . 1 . a^s = b^r a^s.
\]

But \( b^r a^s = b^r a^v \) implies \( u = r, v = s \). Thus \( s < v \) implies \( b^r a^s < b^r a^v \). Likewise \( b^r a^s < b^r a^v \) and \( s - r = v - u \) implies \( s \leq v \) since otherwise, from what we have shown, \( b^r a^v < b^r a^s \). But then \( s = v \) implies \( r = u \) so that \( b^r a^s = b^r a^v \). Hence \( b^r a^s < b^r a^v \) implies \( s < v \).\( \square \)

**Theorem 4.5.** Let \( \leq \) be a compatible partial order on \( B \) with \( 1 < a, ba \). Then the following are equivalent:
(i) the set \( \{ b^n a^n : n \geq 0 \} \) has no upper bound in \( B \);

(ii) for each \( n \geq 0 \), there is an idempotent \( e \in B \) with \( e \neq a^n \);

(iii) for each \( c \in B \), the set \( \{ x \in B : x \leq c \} \) obeys the ascending chain condition;

(iv) the set of elements \( \{ x \in B : x \leq 1 \} \) obeys the ascending chain condition.

**Proof.** Clearly, (i) implies (ii). So suppose (ii) and let \( x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \) be an infinite ascending chain of elements of \( B \) bounded above by \( c \), where \( x_n = b^n a^n \) and \( c = b^r a^s \). Then, since \( \sigma^1 \) is an isotope homomorphism we have

\[
s_1 - r_1 \leq s_2 - r_2 \leq \cdots \leq s_n - r_n \leq \cdots \leq s - r.
\]

Hence, since principal order ideals in \( \mathbb{Z} \) obey the ascending chain condition, there exist \( m, k \) such that \( s_n - r_n = k \) for all \( n > m \). Consequently

\[
1 \leq a^{r_m} x_{m+1} b^{s_m} \leq \cdots \leq a^{r_m} x_{m+n} b^{s_m} \leq \cdots \leq a^{r_m} c b^{s_m} \leq a^{r_m+r} = w,
\]

where each \( e_n = a^{r_m} x_{m+n} b^{s_m} \) is an idempotent. But, since \( s_m - r_m = s_{m+n} - r_{m+n} \), \( x_m \leq x_{m+n} \) implies \( s_m \leq s_{m+n} \) and \( r_m \leq r_{m+n} \). Hence \( e_u = e_v \) implies \( x_{m+u} = b^{r_{m+u}} a^{s_{m+u}} = b^{r_m} a^{r_m} b^{s_{m+u}} a^{s_m + u} b^{s_m} a^{s_m} = b^{r_m} e_u a^{s_m} = b^{r_m} e_v a^{s_m} = x_{m+v} \).

By (ii), there is only a finite number of idempotents less than \( w \). Thus there are only a finite number of distinct members in the set \( x_m \leq \cdots \leq x_n \leq \cdots \leq c \). That is, (iii) holds.

Clearly, also, (iii) implies (iv) so suppose that the set \( \{ b^n a^n : n \geq 0 \} \) has an upper bound \( c = b^r a^s \) in \( B \). Then

\[
1 < ba < b^2 a^2 < \cdots < c = b^r a^s.
\]

Multiplying on the left by \( a^r \) and on the right by \( b^s \) we get

\[
a^r b^s \leq \cdots \leq b^s a^r = a^r b^{r+s} a^{r+s} b^s \leq b^{s+1} a^{r+1} = a^r b^{r+s+1} a^{r+s+1} b^s \leq b^{s+2} a^{r+2} \leq \cdots \leq d^r b^r a^s b^s = 1.
\]

Since \( \{ x \in B : x \leq 1 \} \) obeys the ascending chain condition, it follows that there exists \( n \) such that \( b^{s+n} a^{r+m} = b^{s+n} a^{r+n} \) for all \( m \geq n \). But then, by pre-multiplying by \( a^s \) and post-multiplying by \( b^r \), we find that \( b^n a^n = b^m a^m \) for all \( m \geq n \) which is a contradiction.

Finally, we show that there is exactly one compatible partial order on \( B \) with \( 1 < a, ba \) which does not have a co-atom.
**Theorem 4.6.** Let $\leq$ be a compatible partial order on $B$ with $1 < a, ba$ and suppose that $B$ does not have a co-atom under $\leq$. Then the partial order is given by

$$b^ra^s \leq b^ua^v \text{ if and only if } \begin{cases} 
\text{either } s - r < v - u \\
\text{or } s - r = v - u \text{ and } s \leq v.
\end{cases}$$

Conversely, this partial order turns $B$ into a totally ordered inverse semigroup in which $1 < a, ba$, and which has no co-atom.

**Proof.** The fact that $b^ra^s \leq b^ua^v$ implies either $s - r < v - u$ or $s - r = v - u$ and $s \leq v$ is an immediate consequence of Lemma 4.4.

Conversely, suppose that these relations hold. We show that $b^ra^s \leq b^ua^v$. Suppose first that $s - r < v - u$. Then, since $b^ra^s \leq b^ua^v$ if and only if $b^ra^{s+r} \leq b^ua^{v+r}$ we may assume $s - r > 0$. Then also $v - u > 0$, and $a^{v-u} \leq a^{v-u-1}$ since $s - r < v - u$ implies $s - r \leq v - u - 1$. By the first part of the proof of Proposition 4.3, because $B$ has no co-atom, and $ba > 1$, $b^ra^s < a$ for each $r \geq 0$. Thus

$$b^ra^s = b^ra^s.a^{s-r} \leq b^ra^r.a^{v-u-1} \leq a.a^{v-u-1} = a^{v-u} \leq b^ua^u.a^{v-u} = b^ua^v.$$ 

Now suppose that $s - r = v - u$ and $s \leq v$. If $s \leq r$ then

$$b^ra^s = b^{-s}.b^ra^s \leq b^{-s-u}.b^ua^u = b^ua^v.$$ 

On the other hand, if $s > r$, then, since $s - r = v - u$, $u - r = v - s \geq 0$; so $r \leq u$ and

$$b^ra^s = b^ra^r.a^{s-r} = b^ra^r.a^{v-u} \leq b^ua^u.a^{v-u} = b^ua^v$$

again.

This shows that $\leq$ is as described in the statement of the result. To finish the proof we need only note that $\leq$ is clearly a total order and that, since $0 = r - r < 1 = 1 - 0$, $b^ra^r < b^ra^1 = a$ for all $n \geq 0$. Hence, by Proposition 4.3, since $B$ is, in particular, $\vee$-semilatticed, there is no co-atom.\]

We shall say that $B$ is **lexicographically ordered** if it has the partial order $\leq$ described in the statement of Theorem 4.6 (or one of the other three total orderings in the orbit of $\leq$).

5. **Semilattice Orderings**

Example 2 shows that it is possible for a compatible partial order $\leq$ on $B$, with $1 < a, ba$, to have both a co-atom and have $\{b^na^n : n \geq 0\}$ bounded above. To begin this section we show that this is not possible if $B$ is a $\vee$-semilatticed semigroup under $\leq$. That is, in this case, either the cone for $\leq$ obeys the ascending chain condition, or $b^na^n < a$ for all $n \geq 0$ so that $\leq$ is a total order.
Proposition 5.1. Let ≤ be a compatible partial order on B with $1 < a, ba$ under which $B$ is a $\vee$-semilatticed semigroup. Suppose that $\{b^n a^n : n \geq 0\}$ is bounded above in $B$. Then $b^n a^n < a$ for all $n \geq 0$.

Proof. Let $t = \min\{s \in \mathbb{Z}^+ : b^n a^n < a^s$ for all $n \geq 0\}$. We claim that $t = 1$ which proves the result. Suppose not. Then, by Proposition 4.3, there is a co-atom $b^{r_1} a^r$. Since $b^{n+t} a^{n+t} < a^t$, we have $b^{n+t} a^n < 1$ for all $n \geq 0$. Thus, because $b^{r_1} a^r$ is a co-atom, either $b^{n+t} a^t \leq b^{r_1} a^r$ for all $n \geq 0$ or there exists $n$ such that $b^{n+t} a^t \lor b^{r_1} a^r = 1$.

If the latter occurs, let $m = n + r + 1$. Then

$$b^{n+t} a^n = b^t b^n a^n \leq b^t b^m a^m = b^{m+t} a^m < 1$$

so that $b^{m+t} a^m \lor b^{r_1} a^r = 1$. But then

$$a^m \lor a^{m+t-1} = a^{m+t} (b^{m+t} a^m \lor b^{r_1} a^r) = a^{m+t}.$$ 

Since $m + t > m$ and $m + t > m + t - 1$, this is impossible. Hence $b^{n+t} a^n \leq b^{r_1} a^r$ for all $n \geq 0$.

Let $k \geq 0$ and set $n \geq k + (r + 1) - t$. Then $b^{n+t} a^n \leq b^{r_1} a^r$ implies

$$b^{n+t} a^{n+t} = b^{n+t} a^n a^t \leq b^{r_1} a^r a^t = b^{r_1} a^{r+1} a^{t-1}$$

so that, pre-multiplying by $a^{r+1}$ and post-multiplying by $b^{r+1}$, we get

$$b^{(n+t)-(r+1)} a^{(n+t)-(r+1)} = a^{r+1} b^{n+t} a^{n+t} b^{r+1} \leq a^{r+1} a^{t-1} b^{r+1} = a^{t-1}.$$ 

Hence, since $k \leq n - (r + 1) + t = (n + t) - (r + 1)$, $b^k a^k \leq a^{t-1}$ for all $k \geq 0$. This contradicts the minimality of $t$. Hence $t = 1$. □

Corollary 5.2. Let ≤ be a compatible partial order on $B$ with $1 < a, ba$, under which $B$ is a $\vee$-semilatticed semigroup. Then $B$ is either lexicographically ordered or obeys the ascending chain condition on its cone.

The main results in this section are concerned with situations in which $B$ is a $\vee$-semilatticed semigroup. This is not an accident as the next result shows.

Theorem 5.3. The lexicographic ordering is the only compatible partial ordering ≤ on $B$ with $1 < a, ba$, under which $B$ is a $\wedge$-semilatticed semigroup.

Proof. Suppose that ≤ is a compatible partial ordering on $B$ with $1 < a, ba$ which turns $B$ into a $\wedge$-semilatticed semigroup. Then, from Theorem 4.6, either ≤ is the lexicographic ordering or there is a co-atom..
Suppose the latter. Then there exists $n > 0$ such that $b^na^n < a$ but $b^{n+1}a^{n+1} a$. Then $b^na^n \leq a \land b^{n+1}a^{n+1}$ and, by Lemma 4.2, $a \land b^{n+1}a^{n+1}$ is an idempotent. Hence $a \land b^{n+1}a^{n+1} = b^na^n$. But this implies
\[
1 = a^n.b^n a^n.b^n = a^n(a \land b^{n+1}a^{n+1})b^n = a^n.a.b^n \land a^n.b^{n+1}a^{n+1}.b^n = a \land ba = ba
\]
since $B$ is a $\land$-semilatticed semigroup. This is a contradiction. □

It is known that free groups can be totally ordered but that they admit no lattice orderings which are not total orders; [4; Corollary 24.12, Proposition 24.13]. Theorem 5.3 shows that the bicyclic semigroup shares this property. On the other hand, Sait [9] has shown that no free inverse semigroup can be totally ordered.

**Corollary 5.4.** The bicyclic semigroup $B$ can be totally ordered but it admits no lattice ordering which is not a total ordering.

**Proof.** Suppose that $B$ is a lattice ordered semigroup under a compatible partial ordering $\leq$. Then the same is true under each of the four partial orderings in the orbit of $\leq$. But one of these has the property that 1 is less than both $a, ba$. By Theorem 5.3, $B$ is totally ordered under this ordering and thus under $\leq$. □

Theorem 5.3 shows that there is a unique compatible partial ordering with $1 < a, ba$, which turns $B$ into a $\land$-semilatticed semigroup. However this is not the case for $\lor$-semilatticed orderings. Indeed, as we shall see there is an infinite family of such orderings in addition to the lexicographic ordering.

**Theorem 5.5.** Let $\leq$ be a compatible partial order on $B$ with $1 < a, ba$ under which $B$ is a $\land$-semilatticed semigroup. Then either $B$ is lexicographically ordered or has a unique co-atom $b^na^{n-1}$. In the latter case, the partial order on $B$ is given by
\[
b^ra^r \leq b^ra^v \text{ if and only if } \begin{cases} 
\text{either } s - r \leq v - u \text{ and } s \leq v \\
\text{or } 0 < (n + 1)(s - v) < n(r - u).
\end{cases}
\]

The positive integer $n$ is uniquely determined by the fact that $n = \max\{t : b^ta^t < a\}$.

**Proof.** Suppose that $B$ is not lexicographically ordered under $\leq$ and let $K$ be its cone. Then $B$ has a unique co-atom $b^na^{n-1}$ and $K$ obeys the ascending chain condition. We claim that $K = K(b^na^{n-1})$. The other results follow directly from this.

First of all, note that, since the elements $w$ of $B$ with $w \leq 1$ obey the ascending chain condition and $B$ has a unique co-atom, $w < 1$ implies $w \leq$
$b^na^{n-1}$. Further, by Proposition 3.1, $w = b^r a^s \in K(b^na^{n-1})$ if and only if
$ns \leq (n-1)r$; thus $K(b^na^{n-1}) = \{1, b^na^{n-1}\} \cup \{b^ra^s : ns \leq (n-1)r\}$.

Suppose that $K \neq K(b^na^{n-1})$; we show that this leads to a contradiction. By the ascending chain condition, the set \( \{w \in B : w < 1, w \notin K(b^na^{n-1})\} \) has a maximal member $b^r a^s$. Then $v < u$ and, since $b^r a^s \notin K(b^na^{n-1})$, $(n-1)u < nv$. Hence $v + 1 \leq u$ implies $(n-1)(v + 1) < nv$ so that $n-1 < v$ and so, since $v < u$, $n \leq v < u$. It follows from this that
\[
b^r a^s = b^{u-n} a^{n-1} \cdot a^{v-(n-1)} \leq b^{u-n} a^{v-(n-1)}
\]
since $b^na^{n-1} \leq 1$. Further, since $n > 0$, the inequality above is strict, so that $b^r a^s < b^{u-n} a^{v-(n-1)}$. But $b^r a^s \leq b^na^{n-1}$ implies
\[
b^{u-n} a^{v-(n-1)} = a^n b^r a^s \cdot b^{(n-1)} \leq a^n b^r a^{n-1} a^{n-1} = 1.
\]
Hence, by the maximality of $b^r a^s$, $b^{u-n} a^{v-(n-1)} \in K(b^na^{n-1})$. But
\[
(n-1)(u - n) - n\{v - (n-1)\} = (n-1)u - nv < 0
\]
since $b^r a^s \notin K(b^na^{n-1})$. But this contradicts the fact that $b^{u-n} a^{v-(n-1)} \in
K(b^na^{n-1})$. \( \square \)

We will now show that the converse also holds. Namely, if $K = K(b^na^{n-1})$ for some $n > 0$, then the resulting partial order $\leq$ turns $B$ into a $\lor$-semilatticed inverse semigroup in which $1 < a, ba$. The first step is to identify the upper bound of $b^r a^s$ and $b^s a^v$ under this partial order. It is easy to do this graphically. In fact, it is easy to see graphically that $B$ is a distributive lattice under $\leq$. For, from the form of $\leq$,
\[
b^r a^s \leq b^r a^y \text{ if and only if } \left\{ \begin{array}{l}
y \leq x + (r-s) \text{ and } x \geq s \\
or x < s \text{ and } dy \leq cx + (dr - cs)
\end{array} \right.
\]
where $c = n, d = n-1$. Thus, geometrically, the set of elements $b^r a^s$ of $B$
with $b^r a^x \geq b^r a^s$ corresponds to the set of points, with non-negative integer
coordinates, which lie in the bottom right region bounded by the lines $y = x + (r-s)$ and $dy = cx + (dr - cs)$, which pass through $(s, r)$. Given elements $b^r a^s$ and $b^s a^v$ in general position, it is easy to see that the intersection of
the corresponding regions is given by the bottom right point of the parallelogram
formed by the four lines. However, we need the precise formulae in order to
prove that multiplication distributes over the join operation. Since we have
already seen that $B$ is not a $\land$-semilatticed semigroup under $\leq$, we cannot rely
on the graphical representation alone.

**Lemma 5.6.** Let $K = K(b^da^d)$ where $c - d = 1$ and let $\leq$ be the corresponding
partial order. Then $b^r a^s \lor b^u a^v = b^r a^q$ where
\[
p = c\{(r-s) \land (u-v)\} - (dr - cs) \land (du - cv)
\]
\[
q = d\{(r-s) \land (u-v)\} - (dr - cs) \land (du - cv).
\]
**Proof.** We will find it convenient to use the fact that, for integers $x, y, x \lor y = -( -x \land -y )$.

First we show that $p, q \geq 0$. Since $d \leq c$, we have

$$dr - cs \leq cr - cs = c(r - s) \text{ and } du - cv \leq cu - cv = c(u - v)$$

so that

$$(dr - cs) \land (du - cv) \leq c(r - s) \land (u - v) = c((r - s) \land (u - v))$$

which shows that $p \geq 0$. Also

$$cs - dr \geq ds - dr = d(s - r) \text{ and } cv - du \geq dv - du = d(v - u)$$

so that

$$(cs - dr) \lor (cv - du) \geq d(s - r) \lor d(v - u) = d((s - r) \lor (v - u))$$

which gives

$$-\{(dr - cs) \land (du - cv)\} \geq -d((r - s) \lor (u - v))$$

$$d((r - s) \land (u - v)) \geq (dr - cs) \land (du - cv);$$

that is $q \geq 0$.

Further

$$q - p = (d - c)((r - s) \land (u - v))$$

$$= -((r - s) \land (u - v)) \text{ since } c - d = 1$$

$$= (s - r) \lor (v - u).$$

Thus $q - p \geq s - r$. Next, either $s \leq q$ or $s > q$. In the first case, $b^p a^s \leq b^p a^q$.

In the second, since $r - p \geq s - q$, we also have $r > p$. Further

$$d(r - p) - c(s - q) = (dr - cs) - (dp - cq)$$

$$= (dr - cs) - (c - d)((dr - cs) \land (du - cv))$$

$$= (dr - cs) - (dr - cs) \land (du - cv) \geq 0$$

so that $d(r - p) \geq c(s - q)$ which shows that $b^{r-p} a^{s-q} \in K$. Hence $b^p a^q$ is an upper bound for $b^p a^s$ and similarly for $b^p a^v$.

Conversely, suppose that $b^p a^s, b^p a^v \leq b^p a^x$. Then $x - y \geq s - r, v - u$ so that

$$x - y \geq (s - r) \lor (v - u) = q - p.$$ 

Thus $b^p a^q \leq b^p a^x$ provided that $q \leq x$ so we may suppose that $q > x$ and seek a contradiction. We consider three cases.

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Case (i): $x \geq s \lor v$. We have
\begin{align*}
    d(r - s) &= (dr - cs) + s \leq (dr - cs) + s \lor v \\
    d(u - v) &= (du - cv) + v \leq (du - cv) + s \lor v
\end{align*}
so
\begin{align*}
    q &= d\{(r - s) \land (u - v)\} - (dr - cs) \land (du - cv) \\
    &= d(r - s) \land d(u - v) - (dr - cs) \land (du - cv) \\
    &\leq \{(dr - cs) \land (du - cv) + s \lor v\} - (dr - cs) \land (du - cv) = s \lor v \leq x.
\end{align*}
Thus $b^y a^x \leq b^y a^x$.

Case (ii): $x < s \land v$. Then $b^{r-y}a^{s-x}, b^{u-y}a^{v-x} \in K$ so that $d(r - y) \geq c(s - x)$ and $d(u - y) \geq c(v - x)$ which imply $dr - cs \geq dy - cx, du - cv \geq dy - cx$. Thus
\begin{align*}
    d(p - y) - c(q - x) &= (dp - cq) - (dy - cx) \\
    &= (dr - cs) \land (du - cv) - (dy - cx) \geq 0.
\end{align*}
Hence $b^{r-y}a^{s-x} \in K$ and $b^{u-y}a^{v-x} \leq b^y a^x$.

Case (iii): $s \land v \leq x < s \lor v$. By symmetry, we may suppose that $s < v$ so that $s \leq x < v$ and consequently, since $b^y a^x \leq b^y a^x, d(u - y) \geq c(v - x) > d(u - x)$ so that $u - y > v - x$. Further, since $p - y \geq q - x$ we also have $y < p$ and
\begin{align*}
    d(p - y) - c(q - x) &= (dp - cq) - (dy - cx) \\
    &= (dr - cs) \land (du - cv) - (dy - cx) \geq 0
\end{align*}
if $(dr - cs) \geq (du - cv)$. But $dr - cs < du - cv$ implies
\[0 < c(v - x) \leq d(u - r) \leq c(u - r)\]
so that $v - s \leq u - r$ and $r - s \leq u - v$ whence
\[q = d(r - s) - (dr - cs) = s \leq x.
\]
But this contradicts the assumption that $q > x$. \[\square\]

It remains to prove that under $\leq$ as above, $B$ is a $\lor$-semilatticed semigroup. We do this by showing that the operation $\lor$ is compatible with respect to left and right multiplication by $a, b$. A number of special cases must be dealt with. We consider them in a series of lemmas.

**Lemma 5.7.** The operation $\lor$ is compatible with respect to multiplication on the left by $b$ and on the right by $a$. 

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Proof. We consider multiplication on the left by \( b \); multiplication on the right by \( a \) follows by symmetry.

Suppose that \( b^r a^s \lor b^s a^v = b^p a^q \). Then \( b(b^r a^s \lor b^s a^v) = b^{p+1} a^q \) while \( b^{r+1} a^s \lor b^{v+1} a^v = b^p a^x \) where

\[
y = c\{(r + 1 - s) \land (u + 1 - v)\} - \{(d(r + 1) - cs) \land (d(u + 1) - cv)\} \\
= c\{(r - s) \land (u - v) + 1\} - \{(dr - cs) + d\} \land \{(du - cv) + d\} \\
= c\{(r - s) \land (u - v)\} - (dr - cs) \land (du - cv) + (c - d) \\
= p + 1
\]

and similarly \( x = q \).\( \Box \)

An argument similar to that above shows that, when \( r, u > 0 \), multiplication on the left by \( a \) distributes over \( \lor \). The dual is also true for multiplication on the right by \( b \) whenever \( s, v > 0 \). Furthermore, if \( r = u = 0 \),

\[
a(b^r a^s \lor b^s a^v) = a.a^{s\lor v} = a^{1+(s\lor v)} = a^{1+s} \lor a^{1+v} = a.b^0 a^s \lor a.b^0 a^v
\]

so that, in this case too, distributivity holds. Hence it remains to consider the case where one of \( r, u = 0 \) and the other is not, as well as the dual. We suppose that \( r = 0 \).

Lemma 5.8. Suppose that \( r = 0, u > 0 \). Then \( a(b^r a^s \lor b^u a^v) = a.b^r a^s \lor a.b^u a^v \).

Proof. Since \( r = 0 \), \( b^r a^s \lor b^u a^v = b^p a^q \) where

\[
p = c\{-s \land (u - v)\} - \{(ds) \land (du - cv)\} \\
q = d\{-s \land (u - v)\} - \{(cs) \land (du - cv)\}.
\]

Case(i): \( p = 0 \). Here

\[
a(b^r a^s \lor b^u a^v) = a^{q+1}.
\]

Now \( p = 0 \) if and only if

\[
-cs \land c(u - v) = c\{-s \land (u - v)\} = -cs \land (du - cv)
\]

Since \( du - cv < cu - cv \), the common minimum must be \(-cs\) and equality can hold if and only if \(-cs \leq du - cv \). In this case, \( q = -ds + cs = s \) so that \( a(b^r a^s \lor b^u a^v) = a^{s+1} \).

Further, we then have

\[
a.b^r a^s \lor a.b^u a^v = a^{s+1} \lor b^{u+1} a^v = b^p a^x
\]

where

\[
x = d\{-s + 1 \land (u - 1 - v)\} - \{-c(s + 1) \land (du - d - cv)\} \\
= d\{-s \land (u - v)\} - d - \{-cs \land (du - cv + c - d) - c\} \\
= d(-s) - d - \{(cs) - c\} = (c - d)(s + 1) = s + 1
\]

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since \(du - cv + c - d > du - cv\) implies \(-cs = (-cs) \land (du - cv) = (-cs) \land (du - cv + c - d)\). And, similarly, \(y = 0\). Thus when \(p = 0\), the lemma is also proved.

Case (ii); \(p > 0\). In this case, \(-cs > du - cv\) so that

\[
p = c\{-(s + 1) \land (u - v)\} - (du - cv) \quad \text{and} \quad q = d\{-(s + 1) \land (u - v)\} - (du - cv),
\]

and \(a(b^r a^s \lor b^r a^v) = a.b^r a^q = b^{r-1}a^q\). Further \(a.b^r a^s \lor a.b^r a^v = a^{s+1} \lor b^{r-1}a^v = b^r a^x\) where

\[
y = c\{-(s + 1) \land (u - v)\} - c\{-(cs + d - c) \land (du - cv) - d\}.
\]

Since \(-cs > du - cv\) and \(d - c = -1\), \(-cs + (d - c) \geq du - cv\) so that

\[
y = c\{-(s + 1) \land (u - v)\} - (du - cv) - (c - d) = p - (c - d) = p - 1.
\]

Similarly, \(x = q\). Hence in this case too the lemma is proved. \(\square\)

We have thus proved the following:

**Theorem 5.9.** For each positive integer \(n\), \(B\) is a \(\lor\)-semilatticed inverse semigroup under the partial order defined by

\[
b^r a^s \leq b^r a^x \quad \text{if and only if} \quad \begin{cases} y \leq x + (r - s) \\ \text{and} \quad (n - 1)y \leq nx + \{(n - 1)r - ns\}. \end{cases}
\]

Under this partial order, \(1 < b^r a^n < a\) but \(b^{r+1}a^{n+1} \not\equiv a\).

## 6. Conclusion

As we remarked earlier, we can use the action of \(\mathbb{Z}_2 \times \mathbb{Z}_2\) on the compatible partial orders on \(B\) to obtain all partial orders on \(B\) which make it into a semilatticed semigroup from those in which \(1 < a, ba\). We summarize these results in this section.

**Theorem 6.1.** The bicyclic semigroup \(B\) admits exactly four distinct compatible total orderings, defined as follows:

1. \(b^r a^s \leq b^u a^v\) if and only if either \(s - r < v - u\) or \(s - r = v - u\) and \(s \leq v\);
2. \(b^r a^s \leq b^v a^u\) if and only if either \(s - r > v - u\) or \(s - r = v - u\) and \(s \geq v\);
3. \(b^r a^v \leq b^u a^v\) if and only if either \(s - r > v - u\) or \(s - r = v - u\) and \(s \leq v\);
4. \( b^r a^s \leq b^r a^u \) if and only if either \( s - r < v - u \) or \( s - r = v - u \) and \( s \geq v \).

**Theorem 6.2.** In addition to the four total orders, the bicyclic semigroup \( B \) admits two infinite families of distinct partial orders \( \leq \) which turn \( B \) into a \( \lor \)-semilatticed inverse semigroup. These are defined as follows, for each positive integer \( n \):

\[
\begin{align*}
\quad & b^r a^s \leq b^r a^x \text{ if and only if } y \leq x + (r - s) \text{ and } (n-1)y \leq nx + \{(n-1)r - ns\}; \\
\quad & \text{the positive integer } n = \max \{ s : 1 < b^s a^s < a \}; \\
(i) & b^r a^s \leq b^r a^x \text{ if and only if } y \geq x + (r - s) \text{ and } ny \geq (n-1)x + \{nr - (n-1)s\}; \\
\quad & \text{the positive integer } n = \max \{ s : 1 < b^s a^s < b \}.
\end{align*}
\]

**Theorem 6.3.** In addition to the four total orders, the bicyclic semigroup \( B \) admits two infinite families of distinct partial orders \( \leq \) which turn \( B \) into a \( \land \)-semilatticed inverse semigroup. These are defined as follows, for each positive integer \( n \):

\[
\begin{align*}
(o) & b^r a^s \leq b^r a^x \text{ if and only if } y \geq x + (r - s) \text{ and } (n-1)y \geq nx + \{(n-1)r - ns\}; \\
\quad & \text{the positive integer } n = \max \{ s : a < b^s a^s < 1 \}; \\
(i) & b^r a^s \leq b^r a^x \text{ if and only if } y \leq x + (r - s) \text{ and } ny \leq (n-1)x + \{nr - (n-1)s\}; \\
\quad & \text{the positive integer } n = \max \{ s : b < b^s a^s < 1 \}.
\end{align*}
\]

Recall that a partial order \( \leq \) on an inverse semigroup \( S \) is said to be **left amenable** if \( s \leq t \) implies \( ss^{-1}s \leq t^{-1}t \); it is **right amenable** if \( s \leq t \) implies \( ss^{-1} \leq tt^{-1} \). In the case of the bicyclic semigroup these correspond to \( b^r a^s \leq b^r a^x \) implies \( b^r a^s \leq b^r a^x \) \( b^r a^s \leq b^r a^x \) respectively. Thus, when \( ba > 1 \), \( B \) is left \( [\text{right}] \) amenable if and only if \( b^r a^s \leq b^r a^x \) implies \( s \leq x \) \( [r \leq y] \). When \( ba < 1 \), \( B \) is left \( [\text{right}] \) amenable if and only if \( b^r a^s \leq b^r a^x \) implies \( s \geq x \) \( [r \leq y] \).

**Theorem 6.4.** The bicyclic semigroup \( B \) admits exactly four semilattice orders which are left or right amenable. These are given by:

\[
\begin{align*}
\quad & b^r a^s \leq b^r a^x \text{ if and only if } y \leq x + (r - s) \text{ and } s \leq x; \text{ this is a } \lor \text{-semilattice ordering and is left amenable}; \\
(i) & b^r a^s \leq b^r a^x \text{ if and only if } y \geq x + (r - s) \text{ and } r \leq y; \text{ this is a } \lor \text{-semilattice ordering and is right amenable}; \\
(o) & b^r a^s \leq b^r a^x \text{ if and only if } y \geq x + (r - s) \text{ and } s \geq x; \text{ this is a } \land \text{-semilattice ordering and is left amenable}; \\
(io) & b^r a^s \leq b^r a^x \text{ if and only if } y \leq x + (r - s) \text{ and } r \geq y; \text{ this is a } \land \text{-semilattice ordering and is right amenable}.
\end{align*}
\]
Proof. These orderings are the ones in Theorems 6.2, 6.3 in which \( n = 1 \). The fact that they are left or right amenable follows directly from the definitions.

That they are the only semilatticed orderings with this property follows from those theorems because, if \( n > 1 \), we can find a pair \( b^r a^s \leq b^r a^x \) where \( b^r a^x \not\leq b^r a^y \) and \( b^s a^y \not\leq b^s a^x \). For example, in the case of the first ordering, where \( 1 < a, ba \), we have \( b^2 a^2 < a \) but \( b^2 a^2 \not\leq 1 \) and \( b^2 a^2 \not\leq ba \).

In [6] we dealt with the structure of semilatticed inverse semigroups which are left or right amenable. Particular emphasis was placed on those in which the imposed partial order on the idempotents extends the natural partial order. In the case of the bicyclic semigroup, this means that \( ba < 1 \). Thus the only such left amenable ordering is that given by

\[
(a) \ b^r a^s \leq b^r a^x \text{ if and only if } y \geq x + (r - s) \text{ and } s \geq x.
\]

This is the same as the partial ordering on \( T_E \) in [6, Section 1].

If \( \leq \) is a compatible partial ordering on \( B \) which turns it into a Dubreil-Jacotin semigroup, then the element \( \xi \) must be an idempotent and, indeed, \( \xi = 1 \) so that \( ba < 1 \). If we assume that the partially ordered group \( G \) is the integers under the usual ordering then either \( a\theta = 1 \), so \( b\theta = -1 \), or \( b\theta = 1 \) and \( a\theta = -1 \). Thus either \( b < 1 \), so \( a > 1 \), or \( a > 1 \) and \( b < 1 \). Thus \( \leq \) belongs to the orbit of a partial order in which \( 1 < a, ba \) under the action of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) and so can be analyzed using the results in this paper. Indeed, the conditions for a Dubreil-Jacotin semigroup translate into the following for the partial order with \( 1 < a, ba \):

1. \( (i) \ b^r a^s \geq 1 \) if and only if \( s \geq r \);
   \( (ii) \) for each \( b^r a^s \), the sets \{\( b^r a^x : b^r a^s, b^r a^x \geq 1 \)\} and \{\( b^r a^x : b^r a^x, b^r a^x \geq 1 \)\} are non-empty and equal; further, each has a least element.

By Lemma 2.1, the first of these conditions holds in any partial order in which \( 1 < a, ba \). As for the second, we have the following lemma.

Lemma 6.5. Let \( \leq \) be a compatible partial order on \( B \) in which \( 1 < a, ba \). Then the following are equivalent:

\[
(i) \ b^r a^s, b^r a^x \geq 1; \quad (ii) \ b^y a^s b^r a^x \geq 1; \\
(iii) \ b^{r+y} a^{s+x} \geq 1; \quad (iv) \ b^y a^x \geq b^{s+y+s-x} a^s = a^r b^s.
\]

Proof. By Lemma 2.1, \( b^r a^v \geq 1 \) if and only if \( v \geq u \) if and only if \( (b^s a^v)^{\sigma^1} \geq 1 \). Thus, since the elements on the left side of the first three inequalities in the statement of the lemma have the same images under \( \sigma^1 \), the first three statements are equivalent. Suppose the third. Then \( b^{r+y} a^{s+x} \geq 1 \) implies
\[ b^y a^x = a^r b^{y+r} a^{x+r} b^y b^x \geq a^r 1 b^s = b^{s+r} = b^r a^{s+r-s} \geq b^r b^s \geq 1 \text{ since } ba > 1. \]

It follows that \( B \) is a Dubreil-Jacotin semigroup in which \( \theta \) is an isotone homomorphism onto \( \mathbb{Z} \) with the usual ordering if and only if \( \leq \) satisfies \( ba < 1 < a \) or \( ba, b < 1 \). Thus there exist two total orderings and two infinite families of semilattice orderings under which \( B \) is a Dubreil-Jacotin semigroup.

Finally, we note that the bicyclic semigroup \( B \) can be represented as an \( E \)-unitary semigroup \( P(\mathbb{Z}, \mathbb{Z}^+) \) where \( \mathbb{Z} \) acts on itself, by order automorphisms by \( g.m = -g + m \) and multiplication is given by

\[ (c, g)(d, h) = (c \lor g, d, g + h) = (c \lor (-g + d), g + h). \]

Under this representation, \( a \) corresponds to \((0, 1)\) while \( b \) corresponds to \((1, -1)\).

The partial order corresponding to the cone \( K(b^n a^{n-1}) \) is then given by

\[ (c, g) \leq (d, h) \text{ if and only if } g \leq h \text{ and } ng + c \leq nh + d \]

and

\[ (c, g) \lor (d, h) = ((ng + c) \lor (nh + d) - n(g \lor h), g \lor h). \]

The other members of its orbit are

\[ (c, g) \leq (d, h) \text{ if and only if } g \geq h \text{ and } (n-1)g - c \geq (n-1)h - d \]

\[ (c, g) \leq (d, h) \text{ if and only if } g \geq h \text{ and } ng + c \geq nh + d \]

\[ (c, g) \leq (d, h) \text{ if and only if } g \leq h \text{ and } (n-1)g - c \leq (n-1)h - d \]

respectively.

In the case when \( n = 1 \), the last of these becomes

\[ (c, g) \leq (d, h) \text{ if and only if } g \leq h \text{ and } c \geq d \]

which is closely related to the Cartesian product on \( \mathbb{Z} \times \mathbb{Z} \); the order on the second component is reversed. The Cartesian product ordering

\[ (c, g) \leq (d, h) \text{ if and only if } c \leq d, g \leq h, \]

however, is not compatible with multiplication on \( P(\mathbb{Z}, \mathbb{Z}^+) \).
References


2. B. Bosbach, Zur Struktur der Inversen Teilbarkeitshalbgruppen, Semigroup Forum 12 (1976), 137-144.


