Some Brief Solutions for
Math 232 Practice for Exam #1

1. a) Find parametric equations for the line through the points $P(1,2,3)$ and $Q(-1,5,2)$.

Solution: Start by finding the vector from $P$ to $Q$, which is $<-2,3,-1>$. Then you can use $x = 1 - 2t$, $y = 2 + 3t$ and $z = 3 - t$ for parametric equations.

b) Find the distance from the point $R(2,3,4)$ to the line in part a.

Solution: Start by drawing a line and labeling two points on the line $P$ and $Q$ (the orientation doesn’t really matter for our purposes here, so make $P$ the lefthand point). Draw a point $R$ above the line and draw the vector from $P$ to $R$. Finally draw a line segment which goes perpendicularly from the line up to $R$; we need to find the length of this line segment.

To find this length, notice how we’ve drawn a triangle. If we call the interior angle down by point $P \theta$ and $h$ is the length of the vector from $P$ to $R$, then $h \sin \theta$ is the length we’re after. We can get $\sin \theta$ a couple different ways. One way would be to calculate $\mathbf{a} \times \mathbf{b}$, where $\mathbf{a}$ is the vector from $P$ to $R$ and $\mathbf{b}$ is the vector from $P$ to $Q$. Then

$$\sin \theta = \frac{\left| \mathbf{a} \times \mathbf{b} \right|}{\left| \mathbf{a} \right| \cdot \left| \mathbf{b} \right|}.$$ 

Another way to find $\sin \theta$ is to use the dot product. Since $\mathbf{a} \cdot \mathbf{b} = \left| \mathbf{a} \right| \left| \mathbf{b} \right| \cos \theta$,

$$\sin \theta = \sqrt{1 - \frac{\left| \mathbf{a} \cdot \mathbf{b} \right|^2}{\left| \mathbf{a} \right|^2 \left| \mathbf{b} \right|^2}}.$$ 

This one works out simpler since $\mathbf{a} = <1,1,1>$ is perpendicular to $\mathbf{b} = <-2,3,-1>$.

Any way you work it, $\theta = \pi/2$ and the distance from the line to $R$ is just the distance $h$ from $P$ to $R$, which is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$.

2. a) The points $P$, $Q$ and $R$ above are three vertices of a parallelogram. Find the fourth vertex and the area of this parallelogram.

Solution: The vector which goes from $P$ to $R$ is $<1,1,1>$ and the vector which goes from $P$ to $Q$ is $<-2,3,-1>$. The sum of these two vectors, $<-1,2,0>$, goes from $P$ to the fourth vertex. So the fourth vertex is $(0,6,3)$ The area of the parallelogram is the length of the cross product of $<1,1,1>$ and $<-2,3,-1>$, which is $|<4,1,5>| = \sqrt{16 + 1 + 25} = \sqrt{42}$. 

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b) Find an equation for the plane through $P$, $Q$ and $R$.

Solution: The cross product above, $<-4, 1, 5>$, is a normal vector. Using $P$ is on the plane gives you

$$-4(x - 1) + 1(y - 2) + 5(z - 3) = 0.$$ 

2. Find spherical and cylindrical coordinates for the point with rectangular coordinates $(\sqrt{6}, \sqrt{6}, 2)$.

Solution: The distance $\rho = \sqrt{6 + 6 + 4} = 4$. Since $z = 2$, using $z = \rho \cos \phi$ tells you that $\cos \phi = 1/2$, so $\phi = \pi/3$. Using that and $x = \rho \cos \theta \sin \phi$ tells you that $\cos \theta = \frac{\sqrt{6}}{2\sqrt{3}} = \frac{1}{\sqrt{2}}$, so $\theta = \pm \pi/4$. Since $y$ is positive, it must be the positive angle here.

For cylindrical coordinates, we already have $\theta$ and $z$, so we only need to find $r$. Since $r^2 = x^2 + y^2$, $r = \sqrt{12}$.

3. Describe the traces of the surface given by $x^2 + y^2 - 2z^2 = 0$ in the planes $x = k$, $y = k$ and $z = k$. Use these traces to help sketch this surface.

Solution: Setting $z = k$ gives you $x^2 + y^2 = 2k^2$, which defines a circle no matter what $k$ is. When $k = 0$, however, the circle degenerates to a single point.

Setting $x = k$ gives you $2z^2 - y^2 = k^2$, which defines a hyperbola opening in the $z$ direction except when $k = 0$. In the exceptional case, you get two lines: $\sqrt{2}z - y = 0$ and $\sqrt{2}k + y = 0$. A similar thing happens when you set $y = k$.

This surface consists of two right circular cones which meet at the origin and open along the $z$-axis.

4. Find an equation for the tangent line to the parametric curve $x = \sin(2t)$, $y = e^t$ at the point $(0, 1)$.

Solution: Use $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t}{2 \cos(2t)}$. In order to get the point $x = 0$, $y = 1$, the parameter $t$ must be 0. So the slope is $\frac{e^0}{2 \cos(0)} = 1/2$. An equation for the tangent line is $y - 1 = \frac{1}{2}(x - 0)$.

5. a) Plot at least 12 points by hand and accurately sketch the polar curve $r = 2\sin(3\theta)$. 

Solution: I think you can handle this one on your own.

b) Set up, but do not evaluate, an integral to compute the arclength of the curve in part a. (You do not need to do any “simplifying” of the integrand.)

Solution: Use the fact that this is a parametric curve. You get \( x = r \cos \theta = 2 \sin(3\theta) \cos \theta \) and \( y = r \sin \theta = 2 \sin(3\theta) \sin \theta \). Then \( dx/d\theta = 6 \cos(3\theta) \cos \theta - 2 \sin(3\theta) \sin \theta \) and \( dy/d\theta = 6 \cos(3\theta) \sin \theta + 2 \sin(3\theta) \cos \theta \). The arclength is

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\int_{\alpha}^{\beta} \sqrt{(dx/d\theta)^2 + (dy/d\theta)^2} \, d\theta.
\]

We still need to find the limits of integration. Starting at \( \theta = 0 \), how far must we go before we’ve traced the whole curve? If you try it yourself, you’ll see that you need to go up to \( \theta = \pi \) before you start retracing. So you can use 0 and \( \pi \) for the lower and upper limits of integration, respectively.