Two Induction Proofs

In class on 1/17 I tried to explain how to use induction to prove that $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. In an effort to be clear, I’ve written up such a proof - two in fact - to give you a better idea of how it’s “supposed to be done.”

**Theorem**: For any positive integer $n$,

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

**Proof**: Suppose first that $n = 1$. Then

$$\frac{1(1+1)(2 \cdot 1 + 1)}{6} = \frac{1 \cdot 2 \cdot 3}{6} = \frac{6}{6} = 1 = 1^2.$$

Now assume the statement is true for $n$. Then

$$1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6}$$

$$= \frac{(n+1)(n(2n+1) + 6(n+1))}{6}$$

$$= \frac{(n+1)(2n^2 + 7n + 6)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

$$= \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}.$$

By induction, then, the theorem is true.

Another way to do things is more like I had done in class. The part about $n = 1$ remains the same. For the second part, you can write as follows.

Now assume the statement is true for $n$. Since $2n^2 + 7n + 6 = (n+2)(2n + 3)$,

$$\frac{(n+1)(2n^2 + 7n + 6)}{6} = \frac{(n+1)(n+2)(2n + 3)}{6}$$

$$(*) \quad \frac{(n+1)(2n^2 + 7n + 6)}{6} = \frac{(n+1)(n+1+1)(2(n+1)+1)}{6}.$$
But \(2n^2 + 7n + 6 = n(2n + 1) + 6(n + 1)\) also, so by the induction hypothesis
\[
1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + \frac{6(n+1)^2}{6} \\
= \frac{(n+1)}{6}(n(2n+1) + 6(n+1)) \\
= \frac{(n+1)}{6}(2n^2 + 7n + 6).
\]
By (*) this shows that
\[
1^2 + 2^2 + \cdots + (n+1)^2 = \frac{(n+1)(n+1 + 1)(2(n+1) + 1)}{6}
\]
and the theorem is true by induction.

This second version shows you two important things. First, the “induction hypothesis” is the part of induction when you assume the statement is true for \(n\). Second, it also shows how you may refer back to a previous equation or inequality. If there is something you want to refer to later, such as the equation
\[
\frac{(n+1)(2n^2 + 7n + 6)}{6} = \frac{(n+1)(n+1 + 1)(2(n+1) + 1)}{6},
\]
you “tag” and simply cite the “tag” when you want. Here I tagged it with an asterisk \(\ast\), but you’re free to use any symbol that suits your fancy. If you have more than a couple such tagged equations or inequalities, it’s probably best to label them with numbers.

Here for your reading pleasure is another proof by induction. The result I’ll prove is the familiar binomial theorem. It’s most succinctly stated using the binomial coefficient \(\binom{n}{j}\) which is (in case you’ve forgotten) defined as
\[
\binom{n}{j} = \frac{n!}{j!(n-j)!}.
\]
Remember that \(0! = 1\). I’ll make my proof a little cleaner and more easily followed by first proving an auxiliary result. These are typically labeled as lemmas as opposed to theorems.

**Lemma:** For any two \(n \geq j \in J\),
\[
\binom{n}{j} + \binom{n}{j-1} = \binom{n+1}{j}.
\]
Proof: By definition

\[
\binom{n}{j} + \binom{n}{j-1} = \frac{n!}{j!(n-j)!} + \frac{n!}{(j-1)!(n+1-j)!} = \frac{n!(n+1-j)+n!j}{j!(n+1-j)!} = \frac{n!(n+1)}{j!(n+1-j)!} = \binom{n+1}{j}.
\]

And now for the Binomial Theorem: For any two real numbers \(a\) and \(b\) and any \(n \in J\),

\[
(a + b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j.
\]

Proof: For \(n = 1\),

\[
(a + b)^1 = a + b = (1) a^{1-0} b^0 + (1) a^{1-1} b^1.
\]

Now suppose \(n \geq 1\). By the induction hypothesis

\[
(a + b)^{n+1} = (a + b)^n (a + b) = \sum_{j=0}^{n} \binom{n}{j} a^{n+1-j} b^j + \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^{j+1} = a^{n+1} + \left[\sum_{j=1}^{n} \left(\binom{n}{j} + \binom{n}{j-1}\right) a^{n+1-j} b^j\right] + b^{n+1}.
\]

By the lemma, this shows that

\[
(a + b)^{n+1} = a^{n+1} + \left[\sum_{j=1}^{n} \binom{n+1}{j} a^{n+1-j} b^j\right] + b^{n+1} = \sum_{j=0}^{n+1} \binom{n+1}{j} a^{n+1-j} b^j,
\]

so the binomial theorem is true by induction.

Notice how I used the phrase “by the induction hypothesis” again. Also, notice how I separated the proof of the identity involving the binomial coefficients from the proof of the main result. This is strictly a stylistic choice, but I think it makes the proof “flow” better since the induction proof isn’t interrupted by checking that \(\binom{n}{j} + \binom{n}{j-1} = \binom{n+1}{j}\).
Finally, in this proof I used
\[ \sum_{j=0}^{n-1} \binom{n}{j} a^{n-j} b^j = \sum_{j=1}^{n} \binom{n}{j-1} a^{n+1-j} b^j; \]
some might say I “skipped a step” in my proof. This is always a tough matter. Exactly how much will you assume of the reader? Does the above identity require separate justification, or can I ask the reader to verify it on his/her own? Every proof involves this sort of tradeoff between completeness and conciseness many times. I’m afraid there are no easy answers for you. At least in the beginning of the class, it’s perhaps best to err on the side of completeness. In other words, if I were handing in my proof of the binomial theorem as a class assignment, I’d probably justify this equation. Just so we’re clear on what I mean, here is a more “complete” proof of the binomial theorem.

**Proof:** For \( n = 1 \),
\[ (a + b)^1 = a + b = \binom{1}{0} a^1 b^0 + \binom{1}{1} a^{1-1} b^1. \]
Now suppose \( n \geq 1 \). By the induction hypothesis
\[(*) \quad (a + b)^{n+1} = (a + b)^n (a + b) = \sum_{j=0}^{n} \binom{n}{j} a^{n+1-j} b^j + \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^{j+1}.\]
Now
\[ \sum_{j=0}^{n} \binom{n}{j} a^{n+1-j} b^j = a^{n+1} + \sum_{j=1}^{n} \binom{n}{j} a^{n+1-j} b^j \]
and
\[ \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^{j+1} = \sum_{j=0}^{n-1} \binom{n}{j} a^{n-j} b^{j+1} + b^{n+1}. \]
Letting \( i = j + 1 \), we have
\[ \sum_{j=0}^{n-1} \binom{n}{j} a^{n-j} b^{j+1} = \sum_{i=1}^{n} \binom{n}{i-1} a^{n+1-i} b^i. \]
Thus
\[ \sum_{j=0}^{n} \binom{n}{j} a^{n+1-j} b^j + \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^{j+1} = a^{n+1} + \left[ \sum_{j=1}^{n} \left( \binom{n}{j} + \binom{n}{j-1} \right) a^{n+1-j} b^j \right] + b^{n+1}. \]
By the lemma and \((*)\), this shows that
\[ (a + b)^{n+1} = a^{n+1} + \left[ \sum_{j=1}^{n} \binom{n+1}{j} a^{n+1-j} b^j \right] + b^{n+1} = \sum_{j=0}^{n+1} \binom{n+1}{j} a^{n+1-j} b^j, \]
so the binomial theorem is true by induction.