More on Cycles and Permutations

As noted before, a transposition is a 2-cycle.
Notice how any $m$-cycle can be written as a product (composition) of $m - 1$ transpositions:

$$(i_1, i_2, \ldots, i_m) = (i_1, i_m)(i_1, i_{m-1}) \cdots (i_1, i_2).$$

Now this way of writing a cycle as a product of transpositions is just one way of doing so. There are many different ways.

Examples:

$$(2, 5, 3, 6) = (2, 6)(2, 3)(2, 5) = (5, 2)(3, 5)(6, 3) = (1, 7)(2, 6)(2, 3)(2, 5)(1, 7).$$

$$(1) = (1, 3)(1, 3) = (1, 2)(1, 2)(3, 5)(3, 5).$$

What we want to show is that, while a cycle (or any type of permutation) may be written in different ways as a product of transpositions, and even using different numbers of transpositions, the parity of the number of transpositions used stays the same. In other words, if a permutation can be written as the product of an odd number of transpositions, then it can only be written as the product of an odd number of transpositions. If a permutation can be written as the product of an even number of transpositions, then it can only be written as a product of an even number of transpositions.

How could we do this? How should we think about doing something like this?
What we need is a way to think of permutations as either “even” or “odd.” In other words, we need a suitable function which takes permutations as input and has either 1 or $-1$, say, as output.

Start with $n$ variables: $X_1, X_2, \ldots, X_n$. Take the product

$$\prod_{i<j} (X_i - X_j)$$

and call this product $P$. When $n = 4$, this product is


Generally speaking, there will be $n(n-1)/2$ factors in this product. For every pair of variables $X_i$ and $X_j$, either $X_i - X_j$ or $X_j - X_i$ is a factor of $P$, but not both.

Since the subscripts on the variables are just the numbers 1 through $n$, we can view a permutation as a rearrangement of the variables. In other words, we can view a permutation $\sigma$ as changing the product $P$:

$$P_\sigma = \prod_{i<j} (X_{\sigma(i)} - X_{\sigma(j)}).$$

**Example:** Suppose $n = 4$ again and $\sigma$ is the cycle $(1, 3, 4, 2)$. Then

$$P_\sigma = (X_3 - X_1)(X_3 - X_4)(X_3 - X_2)(X_1 - X_4)(X_1 - X_2)(X_4 - X_2).$$

Just like $P$, the product $P_\sigma$ will have $n(n-1)/2$ factors. For every pair of variables $X_i$ and $X_j$, either $X_i - X_j$ or $X_j - X_i$ will be a factor of $P_\sigma$. In other words, either $P_\sigma$ is $P$, or it is $-P$.

Define the function $f : S_n \to \{1, -1\}$ by the rule

$$P_\sigma = f(\sigma)P.$$
Certainly the identity function won’t change $P$, so $f$ of the identity function is 1. What about the next simplest type of permutation, transpositions?

**Example:** Take $n = 4$ again and suppose $\sigma = (1, 3)$. Then

$$P_\sigma = (X_3 - X_2)(X_3 - X_1)(X_3 - X_4)(X_2 - X_1)(X_2 - X_4)(X_1 - X_4) = -P.$$

$$f(\sigma) = -1.$$

Generally speaking, a transposition $(a, b)$ will only affect those factors involving $X_a$ or $X_b$. Of course, there is one factor involving both of them: $(X_a - X_b)$ (assuming that $a < b$). How many other factors will $(a, b)$ affect? There are $n - 2$ factors involving $X_a$ but not $X_b$, and $n - 2$ factors involving $X_b$ but not $X_a$.

Suppose (as we may) that $a < b$. Of those factors involving $X_a$ but not $X_b$, $(a, b)$ will only change the sign of the factor when the index on the other variable is between $a$ and $b$. But the same can be said of those factors involving $X_b$ but not $X_a$. In other words, the combined effect of the transposition $(a, b)$ on all the factors involving just one of $X_a$ or $X_b$ is nothing, since there are an even number of sign changes.

Of course it’s easy to see the effect of $(a, b)$ on the factor $(X_a - X_b)$: it changes it to $(X_b - X_a)$.

**Theorem:** If $\sigma$ is a transposition, then $f(\sigma) = -1$.

And now for the really crucial step.
Lemma: For any two elements \(\sigma, \tau \in S_n\),

\[ f(\sigma \circ \tau) = f(\sigma)f(\tau). \]

(not really a) Proof: You can, if you like, simply check the four possible cases for \(f(\sigma)\) and \(f(\tau)\). For example, suppose \(f(\sigma) = 1\) and \(f(\tau) = -1\). Then applying \(\tau\) to \(P\) first changes it to \(-P\). Applying \(\sigma\) to \(-P\) won’t change it, since \(\sigma\) doesn’t change \(P\). So the combined effect of the composition \(\sigma \circ \tau\) is to change the sign of \(P\).

Corollary: A permutation \(\sigma \in S_n\) can be written as a product of an even number of transpositions if and only if \(f(\sigma) = 1\). It can be written as a product of an odd number of transpositions if and only if \(f(\sigma) = -1\).

Remark: As mentioned before, an \(m\)-cycle can be written as a product of \(m - 1\) transpositions. Therefore, an \(m\)-cycle is even if and only if \(m\) is odd.

What are the even permutations in \(S_3\)? What are the even permutations in \(S_4\)?

Are exactly half of the permutations in \(S_n\) even?