For our purposes here, we’ll only deal with $2 \times 2$ and $3 \times 3$ matrices, and linear functions from $\mathbb{R}^2$ to $\mathbb{R}^2$ and $\mathbb{R}^3$ to $\mathbb{R}^3$.

Recall that a linear function $L$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is a function of the form

$$ L(x) = Ax + b, $$

where $b = L(0)$ is a (column) vector in $\mathbb{R}^2$ and $A$ is an $2 \times 2$ matrix. The linear functions we’ll look at have $b = 0$. The crucial attribute of such linear functions we need here is that $L(ax + by) = aL(x) + bL(y)$ for any vectors $x$, $y$ and scalars (real numbers) $a$ and $b$.

We need to know how such a function distorts area. By that I mean the following. Take a region in the plane with an area. Put the points in this region into a linear function. What you get out is another (most likely different) region. Is its area related to the area of the original region? The answer is “yes,” and there’s even a simple formula!

To see why, it’s probably best to start with the simplest region we can, the unit square $[0,1] \times [0,1]$. Since we’re only interested in the area, it’s okay to just assume $L(x) = Ax$, since adding a $b$ at the end just translates by the vector $b$, and this doesn’t change area. (Picture a region in the plane, then move it. The area doesn’t change when you move it.)

Let’s look at a specific example.

**Example:**

$$ A = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix}. $$

In this case,

$$ L(i) = A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix}. $$

Similarly, $L(j) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Label these vectors $v_1$ and $v_2$. Any point in the square $[0,1] \times [0,1]$ can be thought of as $a\mathbf{i} + b\mathbf{j}$, where $a$ and $b$ are both in $[0,1]$. Using the “crucial attribute” above, this
linear function does the following to such a point:

\[ L(ai + bj) = aL(i) + bL(j) = av_1 + bv_2. \]

Thus, this linear function \( L \) takes the square to the parallelogram determined by the vectors \( L(i) = v_1 \) and \( L(j) = v_2 \). The area of this parallelogram can be computed using the cross product (think of \( v_1 \) and \( v_2 \) as three dimensional vectors with third coordinate zero). When you sort it all out, \( L \) takes the unit square to a parallelogram with area \( |2 \cdot 1 - 3 \cdot 5| = 13 \).

Hopefully, going through this example helps you realize that, in general, a linear transformation with a coefficient matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) takes the unit square to the parallelogram determined by the two columns of \( A \), which has area \( |ad - bc| \). This motivates the following definition.

**Definition:** The determinant of a 2×2 matrix \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is \( ad - bc \). This is written \( \det(A) \).

**Exercise:** What is the determinant of \( A = \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \)? If \( L(x) = Ax \), what does \( L \) do to the unit square? What happens to area in this case?

It is a fact that the determinant of a product of two matrices is the product of their determinants, i.e., \( \det(AB) = \det(A) \det(B) \). We can use this help figure out more areas.

Suppose you start with a generic parallelogram instead of the unit square. If you apply a linear transformation, how is area distorted? Let’s say our parallelogram is determined by the two vectors \( v_1 \) and \( v_2 \). Now apply the linear transformation \( L(x) = Ax \). Then, just as we did in the example above with the \( i \) and \( j \) vectors, we see that \( L \) takes our starting parallelogram to the parallelogram determined by \( Av_1 \) and \( Av_2 \). What is the area? If \( B \) is the matrix with columns \( v_1 \) and \( v_2 \), then the area of our ending parallelogram is \( |\det(AB)| \), which is \( |\det(A)| \cdot |\det(B)| \). But the area of our starting parallelogram was \( |\det(B)| \), so our linear transformation distorted area by \( |\det(A)| \) again, just like it did to the unit square. In particular, when applied to rectangles, linear transformations distort area by the absolute value of the determinant of the coefficient matrix.

All this can be used to see how changing variables with a linear function changes double integrals. Specifically, suppose \( x = au + bv \) and \( y = cu + dv \), where \( a, b, c \) and \( d \) are real numbers with \( ad - bc \neq 0 \). Let \( L \) be the linear transformation \( L(x) = Ax \), where \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) (so \( L \) takes \( \langle u, v \rangle \) to \( \langle x, y \rangle \)). Then

\[
\iint_{R} f(x, y) \, dx \, dy = \iint_{S} f(au + bv, cu + dv) \cdot |\det(AB)| \, du \, dv,
\]
where $S$ is the region which $\mathbf{L}$ takes to $R$.

How does all this play out in three dimensions? Now we have linear functions with $3 \times 3$ matrices. If $\mathbf{L}(\mathbf{x}) = A\mathbf{x}$ is such a function, then $\mathbf{L}$ will take the unit cube $[0,1] \times [0,1] \times [0,1]$ to the parallelepiped determined by the vectors $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$, which are the columns of $A$. The volume of this parallelepiped is given by the triple scalar product.

**Definition:** The determinant of a $3 \times 3$ matrix

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is

$$\det(A) = a \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg).$$

Again, a linear function with coefficient matrix $A$ distorts volume by $|\det(A)|$, and you get an analogous formula for linear transformations with triple integrals.