Math 620 Fall 2012
Week 11: Galois Theory II

Major Theorems/Results: the Fundamental Theorem of Galois Theory, the Fundamental Theorem of Algebra

Exercises:
1. Prove the Fundamental Theorem of Galois Theory: Let $K$ be a finite Galois extension of $F$. Then
   i) for any $H < G(K, F)$, $H = G(K, K_H)$ and $|H| = [K : K_H]$;
   ii) for any intermediate field $L$ ($F \subseteq L \subseteq K$), $L = K_{G(K,L)}$ and $[K : L] = |G(K,L)|$;
   iii) the correspondence $H \rightarrow K_H$ is a one-to-one correspondence between the subgroups of $G(K, F)$ and the intermediate fields $F \subseteq L \subseteq K$;
   iv) under this correspondence, the normal subgroups of $G(K, F)$ correspond to the normal extensions of $F$ contained in $K$, and if $L$ is such an extension, $G(L, F) \cong G(K, F)/G(K, L)$.

   NOTE: You have everything you need to prove this result. Your proof of each statement shouldn’t be more than a few lines long!

2. If $F$ is a perfect field and $K$ is any finite extension of $F$, prove there are only finitely many intermediate fields $L$ between $F$ and $K$. Can you give a bound on the number of such fields which depends only on $[K : F]$?

3. a) You already showed that the Galois group of $(X^2 + 1)(X^2 - 2)$ over $\mathbb{Q}$ is the Klein four group. Find a splitting field and all intermediate fields, and give the correspondence with the subgroups of the Galois group.
   b) Do the same with $X^4 - 2$ over $\mathbb{Q}$.

4. Find the Galois group of $X^3 - 2$ over $\mathbb{Q}(i)$, where $i \in \mathbb{C}$ satisfies $i^2 = -1$.

5. Here’s a problem from the algebra prelim (qualifying exam) I took: Let $f(X) \in \mathbb{Q}[X]$ be of degree $n > 2$ and let $K$ be a splitting field for $f$ over $\mathbb{Q}$. Suppose $G(K, \mathbb{Q}) = S_n$.
   a) Show that $f$ is irreducible over $\mathbb{Q}$.
   b) If $\alpha \in K$ is a root of $f$, show that $G(\mathbb{Q}(\alpha), \mathbb{Q})$ is trivial (has only one element).
   c) If $n \geq 4$ and $\alpha$ is as above, then $\alpha^n \not\in \mathbb{Q}$. 