**Math 620 Fall 2012**

**Week 5: Rings and Fields**

**Major Definitions:** Ring, division ring, field, vector space over a division ring, linear independence, basis, linear transformation, homomorphism, characteristic, integral domain

**Major Theorems/Results:** any linearly independent set can be extended to a basis, the quotient field of an integral domain

**Exercises:**

1. Let $R$ be a commutative ring with identity of characteristic $p > 0$. The Frobenius map is the function that sends each element of $R$ to its $p$th power. Show that this is a ring homomorphism. (Hint: binomial theorem.)

2. Show that the Frobenius homomorphism of a finite field is an automorphism.

3. Give an example of an infinite field of characteristic $p > 0$ where the Frobenius homomorphism is not an automorphism.

4. Let $\mathbb{F}_q$ be a finite field with $q$ elements containing the finite field $\mathbb{F}_p$ with $p$ elements. Show that $\mathbb{F}_q$ is a vector space over $\mathbb{F}_p$. Conclude that $q$ is a power of $p$.

5. Let $V$ be a vector space of dimension $n$ over a field $\mathbb{F}_q$ with $q < \infty$ elements. How many non-zero elements are in $V$? Show that $\text{GL}_n(\mathbb{F}_q)$ acts transitively on $\mathbb{F}_q^n \setminus \{0\}$. What is the stabilizer of $(1, 0, \ldots, 0) \in \mathbb{F}_q^n$ in $\text{GL}_n(\mathbb{F}_q)$? Use your answers to get a formula for the number of elements in $\text{GL}_n(\mathbb{F}_q)$ in terms of $n$, $q$ and the number of elements in $\text{GL}_{n-1}(\mathbb{F}_q)$. What is the order of $\text{GL}_n(\mathbb{F}_q)$?

6. Show that a finite integral domain is a field (the proof is easier than you think). A much harder result, due to Wedderburn, is that a finite division ring is a field. Maybe we’ll do that later.