Math 620 Fall 2012
Week 8: Algebraic Extensions

**Major Definitions:** algebraic extension, algebraic element, algebraically closed, algebraic closure, degree of an extension

**Major Theorems/Results:** existence of algebraically closed fields and algebraic closures, degrees of extensions in towers

**Exercises:**

1. If \( \alpha \) is algebraic over \( F \) of odd degree, show that \( F(\alpha) = F(\alpha^2) \).

2. Show that \( \mathbb{Q}(i) \) and \( \mathbb{Q}(\sqrt{2}) \) (subfields of \( \mathbb{C} \)) are isomorphic as vector spaces over \( \mathbb{Q} \), but not isomorphic as fields.

3. Explicitly construct an algebraic extension of \( \mathbb{Z}/3\mathbb{Z} \) where the polynomial \( X^3 - 2 \) factors completely. Do the same for \( \mathbb{Z}/5\mathbb{Z}, \mathbb{Z}/7\mathbb{Z} \), and \( \mathbb{Q} \).

4. Suppose \( K_1 \) and \( K_2 \) are algebraic closures of a field \( F \). Prove that there is an isomorphism \( \phi: K_1 \rightarrow K_2 \) such that \( \phi(a) = a \) for all \( a \in F \).

5. Suppose \( p \) and \( q \) are distinct prime numbers. What is \([\mathbb{Q}(\sqrt{p}, \sqrt{q}) : \mathbb{Q}]\)? Justify your answer.