\[ \mathbb{R}, \text{you complete me!} \]

Suppose \( K \) is a field with an absolute value \(| \cdot |\). Given a sequence \( \{a_n\} \subseteq K \), we say the sequence is a **Cauchy sequence** if, for all \( \epsilon > 0 \) there is an \( N \in \mathbb{N} \) with \( |a_n - a_m| < \epsilon \) for all \( n, m \geq N \). We say the sequence is a **null sequence** if, for all \( \epsilon > 0 \) there is an \( N \in \mathbb{N} \) with \( |a_n| < \epsilon \) for all \( n \geq N \). We say the sequence \( a_n \) **converges** to \( a \in K \) if the sequence of differences \( \{a_n - a\} \) is null. Finally, we say two Cauchy sequences \( \{a_n\}, \{b_n\} \) are **equivalent**, and write \( \{a_n\} \sim \{b_n\} \), if the sequence of differences \( \{a_n - b_n\} \) is null. (All those epsilons greater than 0, are they real numbers or rational? Hmmm...)

**Notation:** For any \( c \in K \), we let “\( c \)” (with the quotes) denote the sequence \( f: \mathbb{N} \to K \) defined by \( f(n) = c \) for all \( n \in \mathbb{N} \). In other words, “\( c \)” is the sequence consisting entirely of the number \( c \).

This is clearly a Cauchy sequence for any absolute value on \( K \).

**Lemma:** The equivalence defined above for Cauchy sequences is an equivalence relation.

**Proof:** For any sequence \( \{a_n\} \) we have \( \{a_n - a_n\} = \{0\} \to 0 \), so that \( \{a_n\} \sim \{a_n\} \) for any Cauchy sequence \( \{a_n\} \).

Suppose \( \{a_n\} \sim \{b_n\} \) and let \( \epsilon > 0 \). Then for some \( N \in \mathbb{N} \), \( |a_n - b_n| < \epsilon \) for all \( n \geq N \). Thus \( |b_n - a_n| < \epsilon \) for \( n \geq N \) and \( \{b_n - a_n\} \to 0 \). In other words, \( \{b_n\} \sim \{a_n\} \).

Suppose \( \{a_n\} \sim \{b_n\} \) and \( \{b_n\} \sim \{c_n\} \) and let \( \epsilon > 0 \). Then for some \( N \in \mathbb{N} \), \( |a_n - b_n| < \epsilon/2 \) for all \( n \geq N \) and for some \( M \in \mathbb{N} \), \( |b_n - c_n| < \epsilon/2 \) for all \( n \geq M \). This implies that

\[
|a_n - c_n| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

for all \( n \geq \max\{N, M\} \). Thus \( \{a_n - c_n\} \to 0 \) and \( \{a_n\} \sim \{c_n\} \).

**Exercise 1:** Suppose \( \{a_n\} \) and \( \{b_n\} \) are Cauchy sequences in a field \( K \). Show that the sequences of sums and products, \( \{a_n + b_n\} \) and \( \{a_n b_n\} \), are both Cauchy sequences.

**Exercise 2:** Suppose \( \{a_n\} \) is a Cauchy sequence in a field \( K \). Show that it is bounded, i.e., there is a \( B > 0 \) such that \( |a_n| \leq B \) for all \( n \).

One somehow “expects” all Cauchy sequences to converge. If so, we say the field \( K \) is (topologically) **complete** with respect to the absolute value. This is not the case, in general, though it can be resolved by creating a larger field.

1
Definition: The real numbers \( \mathbb{R} \) is the set of equivalence classes of Cauchy sequences of rational numbers using the usual archimedean absolute value. We’ll write \([\{a_n\}]\) to denote the equivalence class which contains the sequence \(\{a_n\}\). We view \(\mathbb{Q}\) as a subset of \(\mathbb{R}\) by identifying the rational number \(c\) with \([\{c\}]\).

Definition: Define addition and multiplication of real numbers by

\[
[\{a_n\}] + [\{b_n\}] = [\{a_n + b_n\}]
\]

and

\[
[\{a_n\}] \cdot [\{b_n\}] = [\{a_n \cdot b_n\}].
\]

Lemma: These operations are well-defined, i.e., depend only on the equivalence classes and not the particular elements of the equivalence classes used.

Proof: Suppose \(\{a_n\} \sim \{a'_n\}\) and \(\{b_n\} \sim \{b'_n\}\) are all Cauchy sequences. By Exercise 1, both \(\{a_n + b_n\}\) and \(\{a_nb_n\}\) are Cauchy sequences.

Let \(\epsilon > 0\). Then for some \(N_1, M_1 \in \mathbb{N}\), \(|a_n - a'_n| < \epsilon/2\) for all \(n \geq N_1\) and \(|b_n - b'_n| < \epsilon/2\) for all \(n \geq M_1\). This implies that

\[
|(a_n + b_n) - (a'_n + b'_n)| = |(a_n - a'_n) + (b_n - b'_n)| \leq |a_n - a'_n| + |b_n - b'_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

for all \(n \geq \max\{N_1, M_1\}\). Thus, \(\{a_n + b_n\} \sim \{a'_n + b'_n\}\) and addition is well-defined.

By Exercise 2, all four sequences \(\{a_n\}, \{a'_n\}, \{b_n\}\) and \(\{b'_n\}\) are bounded. Thus, there is a \(B > 0\) such that \(|a_n|, |a'_n|, |b_n|, |b'_n| \leq B\) for all \(n \in \mathbb{N}\). For some \(N_2, M_2 \in \mathbb{N}\) we have \(|a_n - a'_n| < \frac{\epsilon}{2B}\) for all \(n \geq N_2\) and \(|b_n - b'_n| < \frac{\epsilon}{2B}\) for all \(n \geq M_2\). This implies that

\[
2|a_nb_n - a'_nb'_n| = |(a_n - a'_n)(b_n + b'_n) + (a_n + a'_n)(b_n - b'_n)|
\]

\[
\leq |(a_n - a'_n)(b_n + b'_n)| + |(a_n + a'_n)(b_n - b'_n)|
\]

\[
= |a_n - a'_n| \cdot |b_n + b'_n| + |a_n + a'_n| \cdot |b_n - b'_n|
\]

\[
\leq |a_n - a'_n| \cdot (|b_n| + |b'_n|) + (|a_n| + |a'_n|) \cdot |b_n - b'_n|
\]

\[
< \frac{\epsilon}{2B} \cdot (2B) + \frac{\epsilon}{2B} \cdot (2B)
\]

\[
= 2\epsilon
\]

for all \(n \geq \max\{N_2, M_2\}\). Thus, \(|a_nb_n - a'_nb'_n| < \epsilon\) for all \(n \geq \max\{N_2, M_2\}\) and \(\{a_nb_n\} \sim \{a'_nb'_n\}\). This shows that multiplication is well-defined.
Theorem 1: The real numbers are a field.

Proof: Since the rational numbers are a field,

\[
(\{a_n\} + \{b_n\}) + \{c_n\} = \{(a_n + b_n) + c_n\} = \{(a_n + (b_n + c_n))\} = \{(a_n) + ((b_n) + \{c_n\})\}
\]

for any Cauchy sequences \(a_n\), \(b_n\) and \(c_n\), and similarly for multiplication. Also,

\[
\{a_n\} + \{b_n\} = \{a_n + b_n\} = \{b_n + a_n\} = \{b_n\} + \{a_n\},
\]

and similarly for multiplication. Next,

\[
((\{a_n\} + \{b_n\}) \cdot \{c_n\}) = \{(a_n + b_n) \cdot c_n\} = \{(a_n \cdot c_n) + (b_n \cdot c_n)\} = \{(a_n) \cdot \{c_n\}) + ((b_n) \cdot \{c_n\})\).
\]

We have

\[
[\text{"0"}] + \{a_n\} = \{0 + a_n\} = \{a_n\}
\]

for any \(\{a_n\} \in \mathbb{R}\). We also have

\[
\{a_n\} + \{-a_n\} = \{a_n + (-a_n)\} = [\text{"0"}]
\]

for any \(\{a_n\} \in \mathbb{R}\), and

\[
\{a_n\} \cdot \{1\} = \{a_n \cdot 1\} = \{a_n\}
\]

for all \(\{a_n\} \in \mathbb{R}\). Clearly "0" \(\not\sim"1"\).

Finally, suppose that \(\{a_n\} \neq [\text{"0"}]\), i.e., \(\{a_n\} \not\sim"0"\). This means that for some \(\epsilon > 0\), there are infinitely many \(n \in \mathbb{N}\) such that \(|a_n| \geq \epsilon\). Since \(\{a_n\}\) is a Cauchy sequence, there is an \(N \in \mathbb{N}\) such that \(|a_n - a_m| < \epsilon/2\) for all \(n, m \geq N\). Since there must be an \(n_0 \geq N\) such that \(|a_{n_0}| \geq \epsilon\), then

\[
|a_m| \geq |a_{n_0}| - |a_{n_0} - a_m| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}
\]

for all \(m \geq N\). In particular, \(a_m \neq 0\) for \(m \geq N\). Further, it is not difficult to see that the sequence \(b_n\) defined by

\[
b_n = \begin{cases} 
a_N & \text{if } n \leq N, 
a_n & \text{if } n \geq N
\end{cases}
\]

is a Cauchy sequence equivalent to \(\{a_n\}\). Thus, we may assume without loss of generality that \(a_n \neq 0\) for all \(n \in \mathbb{N}\). Now

\[
\{a_n\} \cdot \{a_n^{-1}\} = \{a_n \cdot a_n^{-1}\} = [\text{"1"}].
\]
We now extend the absolute value on \( \mathbb{Q} \) to \( \mathbb{R} \) in the obvious way.

**Definition:** For \( \{a_n\} \in \mathbb{R} \), \( ||\{a_n\}|| = ||\{a_n\}|| \).

**Lemma:** This is well-defined, i.e., \( ||\{a_n\}|| \in \mathbb{R} \) and depends only on the equivalence class of \( \{a_n\} \).

**Proof:** Let \( \epsilon > 0 \). Then for some \( N \in \mathbb{N} \) we have \( |a_n - a_m| < \epsilon \) for all \( n, m \geq N \). Since \( |a_n - a_m| \leq |a_n - a_n| \), this implies that \( ||a_n - a_m|| < \epsilon \) for all \( n, m \geq N \) and \( \{a_n\} \) is a Cauchy sequence.

Suppose \( \{a'_n\} \sim \{a_n\} \) and let \( \epsilon > 0 \). Then for some \( N \in \mathbb{N} \), \( |a'_n - a_n| < \epsilon \) for all \( n \geq N \). As above, this implies that \( ||a'_n - a_n|| < \epsilon \) for all \( n \geq N \), so that \( \{a'_n\} \sim \{a_n\} \).

**Definition:** We say a real number \( \{a_n\} \) is greater than zero (or positive), and write \( \{a_n\} > 0 \), if there is an \( N \in \mathbb{N} \) and an \( \epsilon > 0 \) such that \( a_n \geq \epsilon \) for all \( n \geq N \).

**Lemma:** This is well-defined.

**Proof:** Suppose \( \{a'_n\} \) and \( \{a_n\} \) are equivalent Cauchy sequences. Suppose further that there is some \( N \in \mathbb{N} \) and an \( \epsilon > 0 \) such that \( a_n \geq \epsilon \) for all \( n \geq N \). There is an \( M \in \mathbb{N} \) such that \( |a'_n - a_n| < \epsilon/2 \) for all \( n \geq M \). This implies that \( |a'_n| \geq |a_n| - |a'_n - a_n| > \epsilon - \epsilon/2 = \epsilon/2 \) for all \( n \geq M \).

**Definition:** We say a real number \( \{a_n\} \) is greater than a real number \( \{b_n\} \), and write \( \{a_n\} > \{b_n\} \), if \( \{a_n\} - \{b_n\} > 0 \).

**Theorem 2:** The function \( |\cdot| \) on \( \mathbb{R} \) defined above is an (archimedean) absolute value on \( \mathbb{R} \).

**Proof:** Starting with the first property, we have \( |a_n| \geq 0 \) for all \( n \). Suppose that \( \{a_n\} \neq \{0\} \). Then there must be an \( \epsilon > 0 \) such that \( |a_n| \geq \epsilon \) for infinitely many \( n \in \mathbb{N} \). Let \( N \in \mathbb{N} \) be such that \( |a_n - a_m| < \epsilon/2 \) for all \( n, m \geq N \). Since there is an \( n_0 \geq N \) with \( |a_{n_0}| \geq \epsilon \), we have \( |a_m| \geq |a_{n_0}| - |a_{n_0} - a_m| > \epsilon - \epsilon/2 = \epsilon/2 \) for all \( m \geq N \). Thus, \( ||\{a_n\}|| = ||\{a_n\}|| > \{0\} \).

For the second property, using \( |a_n b_n| = |a_n| \cdot |b_n| \) for all \( n \), we have

\[
||\{a_n\} \cdot \{b_n\}|| = ||\{a_n b_n\}|| = ||\{a_n\}|| \cdot ||\{b_n\}|| = ||\{a_n\}|| \cdot ||\{b_n\}||.
\]

Now for the third property - the “triangle inequality.” Suppose first that there is some \( \epsilon > 0 \) and some \( N \in \mathbb{N} \) such that \( |a_n + b_n| + \epsilon \leq |a_n| + |b_n| \) for all \( n \geq N \). Then by the definitions,

\[
||\{a_n\} + \{b_n\}|| = ||\{a_n + b_n\}|| < ||\{a_n + b_n\}|| = ||\{a_n\}|| + ||\{b_n\}||.
\]
So suppose this is not the case and let $\epsilon > 0$. Then there are infinitely many $n \in \mathbb{N}$ such that $|a_n + b_n| + \epsilon/2 > |a_n| + |b_n|$. Also, there is an $N \in \mathbb{N}$ such that $|a_n| - M < \epsilon/6$ for all $n \geq N$. Choose an $n \geq N$ such that $|a_n + b_n| + \epsilon/2 > |a_n| + |b_n|$. Then for all $m \geq N$ we have

$$|a_m| + |b_m| - |a_m + b_m| < |a_n| + \frac{\epsilon}{6} + |b_n| + \frac{\epsilon}{6} - |a_n + b_n| + \frac{\epsilon}{6} = |a_n| + |b_n| - |a_n + b_n| + \frac{\epsilon}{2}$$

$$< \epsilon.$$

By the triangle inequality for rational numbers, $|a_m| + |b_m| \geq |a_m + b_m|$. Thus,

$$|a_m| + |b_m| - |a_m + b_m| < \epsilon$$

for all $m \geq N$ and $\{a_n + b_n\} \sim \{a_n\} + |b_n|$. By the definitions, this means that

$$||\{a_n\} + \{b_n\}| = ||\{a_n\}| + ||\{b_n\}||.$$

**Lemma:** Suppose $\{a_n\}$ and $\{b_n\}$ are two unequal real numbers. Then there is a rational number $c$ such that $|\{a_n\} - [c]| < ||\{a_n\} - \{b_n\}||$.

**Proof:** Since $\{a_n\} \not\sim \{b_n\}$, $\{a_n - b_n\} \not\sim \{c\}$ by Theorem 2, $||\{a_n\} - \{b_n\}| = ||\{a_n - b_n\}|| > 0$ so that there is an $\epsilon > 0$ and an $N \in \mathbb{N}$ such that $|a_n - b_n| \geq \epsilon$ for all $n \geq N$. Also, for some $M \in \mathbb{N}$ we have $|a_n - a_m| < \epsilon/2$ for all $n, m \geq M$. Let $c = a_{N+M}$. Then for all $n \geq N + M$, $|a_n - c| < \epsilon/2$ and $|a_n - b_n| \geq \epsilon$. In particular, $|a_n - b_n| - |a_n - c| \geq \epsilon/2$ for all $n \geq N + M$. By the definitions, this means that $|\{a_n\} - [c]| < ||\{a_n\} - \{b_n\}||$.

**Theorem 3:** The real numbers are a complete field with respect to the archimedean absolute value above.

[Technically speaking, the $\epsilon$ in the definition of Cauchy sequence of real numbers is allowed to be real. But we didn’t have the real numbers yet! Fortunately, it suffices to restrict to the case where $\epsilon$ is a positive rational number by the lemma above.]

**Proof:** Let $\{r_n\}$ be a Cauchy sequence of real numbers. If there is an $r \in \mathbb{R}$ and an $N \in \mathbb{N}$ such that $r_n = r$ for all $n \geq N$ we are done, since in this case $\{r_n\} \to r$. So suppose this is not the
implies that a which we discuss here for completeness (pun intended).

Let \( \epsilon > 0 \). Then there is an \( N \in \mathbb{N} \) such that \( |r_n - r_{m}| < \epsilon/3 \) for all \( n, m \geq N \). By the triangle inequality, we have

\[
|a_n - a_m| = ||"a_n"| - |"a_m"|| = |"a_n"| - r_n + r_n - r_m + r_m - |"a_m"||
\leq |"a_n"| - r_n + |r_n - r_m| + |r_m - |"a_m"||
< |r_n - r_m| + |r_n - r_m| + |r_m - r_m'|
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}
= \epsilon
\]

for all \( n, m \geq N \). This shows that \( \{a_n\} \) is a Cauchy sequence (of rational numbers), i.e., \( \{a_n\} \in \mathbb{R} \).

Let \( \epsilon > 0 \) again. Then there are is an \( N \in \mathbb{N} \) such that \( |r_n - r_{m}|, |a_n - a_m| < \epsilon/3 \) for all \( n, m \geq N \). In particular, \( |r_N - |"a_N"|| < \epsilon/3 \) and also \( |"a_N"| - ||a_m|| < \epsilon/3 \). Using the triangle inequality once more,

\[
|r_n - \{a_m\}| = |r_n - r_N + r_N - |"a_N"| + |"a_N"| - |a_m||
\leq |r_n - r_N| + |r_N - |"a_N"|| + |"a_N"| - |a_m||
< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}
= \epsilon
\]

for all \( n \geq N \). Thus, \( \{r_n\} \rightarrow \{a_m\} \in \mathbb{R} \).

The real numbers aren’t just any old field, however. They also satisfy additional properties which we discuss here for completeness (pun intended).

**Lemma:** The sum and product of two positive real numbers is positive.

**Proof:** Suppose \( \{a_n\} \) and \( \{b_n\} \) are positive real numbers. Then by definition there are \( N_1, N_2 \in \mathbb{N} \) and \( \epsilon_1, \epsilon_2 > 0 \) such that \( a_n \geq \epsilon_1 \) for all \( n \geq N_1 \) and \( b_n \geq \epsilon_2 \) for all \( n \geq N_2 \). This implies that \( a_n + b_n \geq \epsilon_1 + \epsilon_2 > 0 \) and \( a_n b_n \geq \epsilon_1 \epsilon_2 > 0 \) for all \( n \geq \max\{N_1, N_2\} \). In other words, \( \{a_n\} + \{b_n\} = \{a_n + b_n\} > 0 \) and \( \{a_n\} \cdot \{b_n\} = \{a_n b_n\} > 0 \).

**Theorem 4:** The real numbers are a totally ordered field.

**Proof:** Suppose \( x < y \). By definition, this means \( y - x > 0 \). Since \((y + z) - (x + z) = y - x\), we have \( y + z > x + z \). Suppose \( x < y \) and \( y < z \). Then \( z - x = (z - y) + (y - x) \) is positive, being
the sum of two positive real numbers. Suppose $x < y$ and $z > 0$. Then $z(y - x)$ is positive, so that $zy > zx$.

Suppose $x \not< y$ and $y \not< x$, and write $x = \{a_n\}, y = \{b_n\}$. Let $\epsilon > 0$. There are infinitely many $n \in \mathbb{N}$ such that $a_n - b_n < \epsilon/3$ and infinitely many $n \in \mathbb{N}$ such that $b_n - a_n < \epsilon/3$. Also, there are $N, M \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/6$ for all $n, m \geq N$ and $|b_n - b_m| < \epsilon/6$ for all $n, m \geq M$. Choose an $n_0, m_0 \geq \max\{N, M\}$ such that $a_{n_0} - b_{n_0} < \epsilon/3$ and $b_{m_0} - a_{m_0} < \epsilon/3$. Then $b_{n_0} - a_{n_0} < b_{m_0} + \epsilon/6 - a_{m_0} + \epsilon/6 < 2\epsilon/3$. For any $n \geq \max\{N, M\}$ we have

$$|a_n - b_n| = |a_n - a_{n_0} + a_{n_0} - b_{n_0} + b_{n_0} - b_n|$$

$$\leq |a_n - a_{n_0}| + |a_{n_0} - b_{n_0}| + |b_{n_0} - b_n|$$

$$< \frac{\epsilon}{3} + \frac{2\epsilon}{3} + \frac{\epsilon}{6}$$

$$= \epsilon.$$ 

Thus $\{a_n - b_n\} \rightarrow 0$ and $\{a_n\} = \{b_n\}$. This shows that either $x < y$ or $y < x$ or $x = y$.

Finally, suppose $x > y$ and $y < x$. Then $[ "0" ] = (x - y) + (y - x)$ is positive since it is the sum of two positive numbers. Similarly, if $x = y$ and either $x > y$ or $y > x$, then $[ "0" ]$ is positive. But clearly $[ "0" ]$ is not positive, so that at most one of $x > y$, $y > x$ and $x = y$ holds.

**Theorem 5:** The real numbers satisfy the least upper bound property.

[This is really a corollary of theorems 3 and 4 above, and isn’t really useful by itself once we have them.]

**Proof:** Suppose $S \subseteq \mathbb{R}$ is a non-empty subset bounded above, and let $B$ be an upper bound for the elements of $S$. Let $s \in S$.

Let $m_0 \in \mathbb{N}$ be largest such that $B - m_0$ is an upper bound for $S$. Then $m_0$ exists since $B - 0$ is an upper bound for $S$ and $B - n$ is not an upper bound for $S$ whenever $n > B - s$.

We claim that there is a sequence $B_1, B_2, \ldots$ of upper bounds for $S$ such that $B_n - 2^{-n}$ is not an upper bound for $S$ and $0 \leq B_{n+1} - B_n \leq 2^{-n}$ for all $n \in \mathbb{N}$. We prove this claim by induction.

Let $m_1 \in \mathbb{N}$ be largest such that $B_0 - m_1 2^{-1}$ is an upper bound for $S$. Then $m_1$ is either 0 or 1 since $B_0 - 0$ is an upper bound for $S$ and $B_0 - 1 = B_0 - 2 \cdot 2^{-1}$ is not. Let $B_1 = B_0 - m_1 2^{-1}$. Then $B_1$ is an upper bound for $S$ and $0 \leq m_1 2^{-1} = B_0 - B_1 \leq 2^{-1}$. Also, $B_1 - 2^{-1} = B_0 - (m_1 + 1) 2^{-1}$ in not an upper bound for $S$ by the definition of $m_1$.

Now suppose $B_1, \ldots, B_n$ have been chosen which satisfy the requirements above. Let $m_{n+1}$
be the largest integer such that \( B_n - m_{n+1}2^{-n-1} \) is an upper bound for \( S \). By the induction hypothesis, \( B_n \) is an upper bound for \( S \) and \( B_n - 2^{-n} \) is not, so that \( m_{n+1} \) is either 0 or 1. Let \( B_{n+1} = B_n - m_{n+1}2^{-n-1} \). Then \( B_{n+1} \) is an upper bound for \( S \) and \( 0 \leq m_{n+1}2^{-n-1} = B_n - B_{n+1} \leq 2^{-n-1} \). Finally, \( B_{n+1} - 2^{-n-1} = B_n - (m_{n+1}+1)2^{-n-1} \) is not an upper bound for \( S \) by the definition of \( m_{n+1} \).

We claim that this is a Cauchy sequence. Indeed, let \( \epsilon > 0 \) and suppose \( n, m \in \mathbb{N} \) with \( n > m \). Then

\[
B_n - B_m = (B_n - B_{n-1}) + \ldots + (B_{m+1} - B_m) \\
\leq 2^{-n} + \ldots + 2^{-m-1} \\
= 2^{-m-1}(1 + \ldots + 2^{m+1-n}) \\
= 2^{-m-1} \frac{1 - 2^{m-n}}{1 - 2^{-1}} \\
= 2^{-m}(1 - 2^{m-n}) \\
< 2^{-m}.
\]

Thus, if \( n, m \geq N \) where \( 2^{-N} \leq \epsilon \), then \( |B_n - B_m| < \epsilon \).

Let \( B_\infty \) be the limit of this Cauchy sequence. Let \( r \in S \). Suppose \( r > B_\infty \) and let \( \epsilon = r - B \). For some \( n \in \mathbb{N} \), \( |B_n - B_\infty| < \epsilon \). But this implies that \( r - B_\infty > |B_n - B_\infty| \), so that \( r > B_n \). This contradicts the fact that all \( B_n \)'s are upper bounds for \( S \). Thus, \( B_\infty \) is an upper bound for \( S \).

Finally, suppose \( B' \) is an upper bound for \( S \). If \( B' < B_\infty \), then \( B_\infty - B' \geq 2^{-n_0} \) for some \( n_0 \in \mathbb{N} \), which implies that \( B_\infty - 2^{-n_0} \) is an upper bound for \( S \). Choose \( N \in \mathbb{N} \) such that \( |B_n - B_\infty| < 2^{-n_0-1} \) for all \( n \geq N \). We then have \( B_n - 2^{-n_0-1} > B_\infty - 2^{-n_0} \) for all \( n \geq N \). But \( B_n - 2^{-n} \) is not an upper bound for any \( n \), so that \( B_n - 2^{-n_0-1} \leq B_n - 2^{-n} \) is not an upper bound for any \( n > n_0 \). This contradiction shows that \( B' \geq B \), so that \( B \) is a least upper bound for \( S \).

**Exercise 3:** Do whatever is necessary to convince yourself that the process carried out above for \( \mathbb{Q} \) and the archimedean absolute value can be used for any absolute value on a field \( K \) to get a topological completion \( \overline{K} \). In particular, it can be used for \( \mathbb{Q} \) and the \( p \)-adic absolute values \( | \cdot |_p \) to get the \( p \)-adic numbers, \( \mathbb{Q}_p \).

**Exercise 4:** Let \( K \) be a field with an absolute value \( | \cdot | \). Complete \( K \) as above and, using the notation established above for elements of \( \mathbb{R} \), define \( \phi: K \to \overline{K} \) by \( \phi(a) = \left[ \frac{a}{n} \right] \). Show that this is an isomorphism of \( K \) into \( \overline{K} \) (so that \( K \) is isomorphically contained in the completion \( \overline{K} \)).