We fix a number field $K$. The degree of $K$ over $\mathbb{Q}$ is denoted by $n$. There are $n$ embeddings $\sigma: K \to \mathbb{C}$; there are $r$ embeddings into $\mathbb{R}$ and $s$ pairs of complex conjugate embeddings into $\mathbb{C}$ (not real). Thus $n = r + 2s$. These embeddings are ordered so that $\sigma_i: K \to \mathbb{R}$ for $i \leq r$ and $\sigma_{i+s} = \overline{\sigma_i}$ for $r + 1 \leq i \leq r + s$, where the overline denotes complex conjugation. As usual $\Delta_K$, denotes the square root of the absolute value of the discriminant of $K$. We also use $e_i = \begin{cases} 1 & \text{if } i \leq r, \\ 2 & \text{if } r + 1 \leq i \leq r + s. \end{cases}$

Define $\rho: K \to \mathbb{R}^n$ by

$$\rho(\alpha) = \left( \sigma_1(\alpha), \ldots, \sigma_r(\alpha), \Re(\sigma_{r+1}(\alpha)), \ldots, \Re(\sigma_{r+s}(\alpha)), \Im(\sigma_{r+1}(\alpha)), \ldots, \Im(\sigma_{r+s}(\alpha)) \right).$$

Note that we previously proved the following result.

**Proposition 1:** Let $\mathfrak{A}$ be a non-zero fractional ideal of $K$. Then $\rho(\mathfrak{A})$ is a lattice in $\mathbb{R}^n$ with

$$\det(\rho(\mathfrak{A})) = N(\mathfrak{A})2^{-s}\Delta_K.$$

We set

$$(\mathbb{R}^n)^* = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \cdots x_r(x_{r+1}^2 + x_{r+s+1}^2) \cdots (x_{r+s}^2 + x_{r+2s}^2) \neq 0 \}$$

and

$$H = \{ y = (y_1, \ldots, y_{r+s}) \in \mathbb{R}^{r+s} : y_1 + \cdots y_{r+s} = 0 \},$$

so that $H \subset \mathbb{R}^{r+s}$ is a subspace of dimension $r + s - 1$. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ set

$$p(x) := x_1 \cdots x_r(x_{r+1}^2 + x_{r+s+1}^2) \cdots (x_{r+s}^2 + x_{r+2s}^2).$$

(Note that $p \circ \rho(\alpha) = N_K/\mathbb{Q}(\alpha)$ for all $\alpha \in K$.) Define the map $pr: (\mathbb{R}^n)^* \mapsto H$ by

$$pr(x_1, \ldots, x_n) = \left( \log |x_1|, \ldots, \log |x_r|, \log(x_{r+1}^2 + x_{r+s+1}^2), \ldots, \log(x_{r+s}^2 + x_{r+2s}^2) \right) - \left( (e_1/n) \log |p(x)|, \ldots, (e_{r+s}/n) \log |p(x)| \right).$$

We also previously proved the following result.

**Proposition 2:** Let $U \subset \mathcal{O}_K^\times$ denote the group of units of the ring $\mathcal{O}_K$. Then $pr \circ \rho(U)$ is a lattice in $H$ of dimension $r + s - 1$. Thus, there are units $\epsilon_1, \ldots, \epsilon_{r+s-1}$ such that any unit $\epsilon$ may be written uniquely as a product

$$\epsilon = \nu^{n_1} \cdots \nu^{n_{r+s-1}},$$

where $\nu$ is a root of unity and $n_1, \ldots, n_{r+s-1} \in \mathbb{Z}$.

For a fixed bound $B > 0$, set

$$C(B) := \{ x \in \mathbb{R}^n : |p(x)| \leq B \}.$$

Let $\epsilon_1, \ldots, \epsilon_{r+s-1}$ be a system of fundamental units as in Proposition 2, so that

$$\mathcal{F} := \{ y = c_1 pr \circ \rho(\epsilon_1) + \cdots + c_{r+s-1} pr \circ \rho(\epsilon_{r+s-1}) : c_1, \ldots, c_{r+s-1} \in [0, 1] \}$$
is a fundamental domain for the lattice \( \text{pr} \circ \rho(U) \subset H \) in Proposition 2. Set \( C' \) to be the inverse image

\[
C' := \text{pr}^{-1}(\mathcal{F}).
\]

Thus, each non-zero principal ideal \( I = (\alpha) \subseteq \mathcal{O}_K \) will have precisely \( w_K \) representatives in the intersection \( \rho(\mathcal{O}_K) \cap C(B) \cap C' \), where \( w_K \) denotes the number of roots of unity in \( \mathcal{O}_K \). One notes from the definitions that \( C(B) = B^{1/n}C(1) \) and \( B^{1/n}C' = C' \) for all \( B > 0 \). Thus, the volume of \( C(B) \cap C' \) is \( BV \), where \( V \) denotes the volume of \( C(1) \cap C' \). Our goal is to compute \( V \).

We first remark that the statement \((x_1, \ldots, x_n) \in C_1(1) \cap C'\) depends entirely on the quantities \(|x_i|\) for \( 1 \leq i \leq r \) and \( \sqrt{x_i^2 + x_{i+s}^2} \) for \( r + 1 \leq i \leq r + s \). Letting

\[
u_i = \begin{cases} |x_i| & \text{if } 1 \leq i \leq r, \\ \sqrt{x_i^2 + x_{i+s}^2} & \text{if } 1 + r \leq i \leq r, \end{cases}
\]

and

\[
u = \prod_{i=1}^{r+s} \nu_i^{e_i},
\]

we have

\[
V = 2^r (2\pi)^s \int \cdots \int_{D} \prod_{i=1}^{r+s} \nu_i^{e_i-1} du_i,
\]

where the domain of integration \( D \) is defined by

\[
u < 1, \ \nu_i > 0 \text{ for } i = 1, \ldots, r + s,
\]

and

\[
(e_1 \log \nu_1 - (e_1/n) \log \nu, \ldots, e_{r+s} \log \nu_{r+s} - (e_{r+s}/n) \log \nu) \in \mathcal{F}.
\]

Letting \( v_i = \nu_i^{e_i} \) gives

\[
V = 2^r \pi^s \int \cdots \int_{D'} \prod_{i=1}^{r+s} dv_i,
\]

where the domain of integration \( D' \) is defined by

\[
u < 1, \ \nu_i > 0 \text{ for } i = 1, \ldots, r + s,
\]

and

\[
(\log v_1 - (e_1/n) \log u, \ldots, \log v_{r+s} - (e_{r+s}/n) \log u) \in \mathcal{F}.
\]

Note that \( D' \) is given by \( 0 < \nu < 1 \) and \( c_1, \ldots, c_{r+s-1} \in [0, 1) \), where

\[
(\log v_1 - (e_1/n) \log u, \ldots, \log v_{r+s} - (e_{r+s}/n) \log u) = c_1 \text{pr} \circ \rho(e_1) + \cdots + c_{r+s-1} \text{pr} \circ \rho(e_{r+s-1}).
\]

Thus

\[
V = 2^r \pi^{s} \int_{0}^{1} \cdots \int_{0}^{1} \frac{\partial(v_1, \ldots, v_{r+s})}{\partial(u, c_1, \ldots, c_{r+s-1})} |du| \prod_{j=1}^{r+s-1} dc_j.
\]

It remains to compute the Jacobian.
To do that, we note that
\[
\log v_i = \frac{e_i}{n} \log u + e_i \sum_{j=1}^{r+s-1} c_j \log |\sigma_i(\epsilon_j)|
\]
for all \(i = 1, \ldots, r + s\). Via logarithmic differentiation, this implies that
\[
\frac{\partial v_i}{\partial u} = \frac{e_i v_i}{nu}
\]
for all \(i = 1, \ldots, r + s\) and
\[
\frac{\partial v_i}{\partial c_j} = v_i e_i \log |\sigma_i(\epsilon_j)|
\]
for all \(i = 1, \ldots, r + s\) and \(j = 1, \ldots, r + s - 1\). The Jacobian is thus
\[
\det \begin{pmatrix}
  e_1 v_1 \log |\sigma_1(\epsilon_1)| & \cdots & e_{r+s} v_{r+s} \log |\sigma_{r+s}(\epsilon_1)| \\
  \vdots & \ddots & \vdots \\
  e_1 v_1 \log |\sigma_1(\epsilon_{r+s-1})| & \cdots & e_{r+s} v_{r+s} \log |\sigma_{r+s}(\epsilon_{r+s-1})|
\end{pmatrix}
\]
\[
= \frac{1}{n} \det \begin{pmatrix}
  e_1 \log |\sigma_1(\epsilon_1)| & \cdots & e_{r+s} \log |\sigma_{r+s}(\epsilon_1)| \\
  \vdots & \ddots & \vdots \\
  e_1 \log |\sigma_1(\epsilon_{r+s-1})| & \cdots & e_{r+s} \log |\sigma_{r+s}(\epsilon_{r+s-1})|
\end{pmatrix}
\]
\[
= \frac{1}{n} \det \begin{pmatrix}
  e_1 \log |\sigma_1(\epsilon_1)| & \cdots & e_{r+s-1} \log |\sigma_{r+s-1}(\epsilon_1)| & 0 \\
  \vdots & \ddots & \vdots & \vdots \\
  e_1 \log |\sigma_1(\epsilon_{r+s-1})| & \cdots & e_{r+s-1} \log |\sigma_{r+s-1}(\epsilon_{r+s-1})| & 0 \\
  e_1 & \cdots & e_{r+s-1} & n
\end{pmatrix}
\]
\[
= R_K,
\]
the regulator of \(K\). (The last column in the latter matrix above is the sum of the columns of the previous matrix.)

We thus have
\[
V = 2^r \pi^s \int_0^1 \cdots \int_0^1 R_K \, du \prod_{j=1}^{r+s-1} dc_j = 2^r \pi^s R_K.
\]