The Real Numbers

Here we show one way to explicitly construct the real numbers \( \mathbb{R} \). First we need a definition.

**Definitions/Notation**: A sequence of rational numbers is a function \( f : \mathbb{N} \to \mathbb{Q} \). Rather than write \( f \) and \( f(n) \) for the sequence and its values, we typically write \( \{a_n\} \) and \( a_n \). A sequence \( \{a_n\} \) is called a Cauchy sequence if, for all \( \epsilon > 0 \) (which we may take to be a rational epsilon for now), there is an \( N \in \mathbb{N} \) such that \( |a_n - a_m| < \epsilon \) for all \( n, m \geq N \). For any rational number \( c \), we let “c” (with the quotes) denote the sequence \( \{a_n\} \) defined by \( a_n = c \) for all \( n \in \mathbb{N} \). In other words, “c” is the sequence consisting entirely of the number \( c \). This is clearly a Cauchy sequence. We say a sequence \( \{a_n\} \) converges to the (rational) number \( c \), and write \( \{a_n\} \to c \), if for all \( \epsilon > 0 \), there is an \( N \in \mathbb{N} \) such that \( |a_n - c| < \epsilon \) for all \( n \geq N \).

**Lemma**: If \( \{a_n\} \) and \( \{b_n\} \) are Cauchy sequences, then so are \( \{a_n + b_n\} \) and \( \{a_n b_n\} \). All Cauchy sequences are bounded.

**Proof**: Let \( \epsilon > 0 \). By definition, there are \( N_1, N_2 \in \mathbb{N} \) such that \( |a_n - a_m| < \epsilon/2 \) for all \( n, m \geq N_1 \) and \( |b_n - b_m| < \epsilon/2 \) for all \( n, m \geq N_2 \). Let \( N = \max\{N_1, N_2\} \). Then for all \( n, m \geq N \) we have

\[
|a_n + b_n| - |a_m + b_m| = |(a_n - a_m) + (b_n - b_m)| \\
\leq |a_n - a_m| + |b_n - b_m| \\
< \epsilon/2 + \epsilon/2 \\
= \epsilon.
\]

This shows that \( \{a_n + b_n\} \) is a Cauchy sequence.

Next, get \( N_3, N_4 \in \mathbb{N} \) such that \( |a_n - a_m| < 1 \) for all \( n, m \geq N_3 \) and \( |b_n - b_m| < 1 \) for all \( n, m \geq N_4 \). Set \( B_1 = \max_{n \leq N_3} \{|a_n| + 1\} \) and \( B_2 = \max_{n \leq N_4} \{|b_n| + 1\} \). One readily verifies that \( |a_n| < B_1 \) and \( |b_n| < B_2 \) for all \( n \in \mathbb{N} \) (this shows that all Cauchy sequences are bounded). Note that \( B_1, B_2 > 0 \) by construction. Now there are \( N_5, N_6 \in \mathbb{N} \) such that \( |a_n - a_m| < \epsilon/(2B_2) \) for all \( n, m \geq N_5 \) and \( |b_n - b_m| < \epsilon/(2B_1) \) for all \( n, m \geq N_6 \). Let \( N' = \max\{N_5, N_6\} \). Then for all
\[ n, m \geq N' \text{ we have} \]
\[ |a_n b_n - a_m b_m| = |a_n(b_n - b_m) + b_m(a_n - a_m)| \]
\[ \leq |a_n(b_n - b_m)| + |b_m(a_n - a_m)| \]
\[ = |a_n| \cdot |b_n - b_m| + |b_m| \cdot |a_n - a_m| \]
\[ < B_1 \cdot \epsilon/(2B_1) + B_2 \cdot \epsilon/(2B_2) \]
\[ = \epsilon. \]

This shows that \( \{a_n b_n\} \) is a Cauchy sequence.

**Definition:** We say two Cauchy sequences \( \{a_n\} \) and \( \{b_n\} \) of rational numbers are equivalent, and we write \( \{a_n\} \sim \{b_n\} \), if the sequence \( \{a_n - b_n\} \to 0 \).

**Lemma:** This is an equivalence relation.

**Proof:** For any sequence \( \{a_n\} \) of rational numbers, \( \{a_n - a_n\} = \text{“0”} \to 0 \), so that \( \{a_n\} \sim \{a_n\} \) for any Cauchy sequence \( \{a_n\} \).

Suppose \( \{a_n\} \sim \{b_n\} \) and let \( \epsilon > 0 \). Then for some \( N \in \mathbb{N} \), \( |a_n - b_n| < \epsilon \) for all \( n \geq N \). Thus \( |b_n - a_n| < \epsilon \) for \( n \geq N \) and \( \{b_n - a_n\} \to 0 \). In other words, \( \{b_n\} \sim \{a_n\} \).

Suppose \( \{a_n\} \sim \{b_n\} \) and \( \{b_n\} \sim \{c_n\} \) and let \( \epsilon > 0 \). Then for some \( N \in \mathbb{N} \), \( |a_n - b_n| < \epsilon/2 \) for all \( n \geq N \) and for some \( M \in \mathbb{N} \), \( |b_n - c_n| < \epsilon/2 \) for all \( n \geq M \). This implies that
\[ |a_n - c_n| = |a_n - b_n + b_n - c_n| \leq |a_n - b_n| + |b_n - c_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \]

for all \( n \geq \max\{N, M\} \). Thus \( \{a_n - c_n\} \to 0 \) and \( \{a_n\} \sim \{c_n\} \).

**Definition:** The real numbers \( \mathbb{R} \) is the set of equivalence classes of Cauchy sequences of rational numbers. We’ll write \( \{\{a_n\}\} \) to denote the equivalence class which contains the sequence \( \{a_n\} \). We view \( \mathbb{Q} \) as a subset of \( \mathbb{R} \) by identifying the rational number \( c \) with \( \{c\} \).

**Definition:** Define addition and multiplication of real numbers by
\[ \{a_n\} + \{b_n\} = \{a_n + b_n\} \]
and
\[ \{a_n\} \cdot \{b_n\} = \{a_n \cdot b_n\} \]

**Lemma:** These operations are well-defined, i.e., depend only on the equivalence classes and not the particular elements of the equivalence classes used.
Proof: Suppose \( \{a_n\} \sim \{a'_n\} \) and \( \{b_n\} \sim \{b'_n\} \) are all Cauchy sequences. By the lemma above, both \( \{a_n + b_n\} \) and \( \{a_n b_n\} \) are Cauchy sequences.

Let \( \epsilon > 0 \). Then for some \( N_1, M_1 \in \mathbb{N} \), \( |a_n - a'_n| < \epsilon/2 \) for all \( n \geq N_1 \) and \( |b_n - b'_n| < \epsilon/2 \) for all \( n \geq M_1 \). This implies that

\[
| (a_n + b_n) - (a'_n + b'_n) | = | (a_n - a'_n) + (b_n - b'_n) | \leq | a_n - a'_n | + | b_n - b'_n | < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\]

for all \( n \geq \max\{N_1, M_1\} \). Thus, \( \{a_n + b_n\} \sim \{a'_n + b'_n\} \) and addition is well-defined.

By a previous lemma, all four sequences \( \{a_n\}, \{a'_n\}, \{b_n\} \) and \( \{b'_n\} \) are bounded. Thus, there is a \( B > 0 \) such that \( |a_n|, |a'_n|, |b_n|, |b'_n| \leq B \) for all \( n \in \mathbb{N} \). For some \( N_2, M_2 \in \mathbb{N} \), we have \( |a_n - a'_n| < \frac{\epsilon}{2B} \) for all \( n \geq N_2 \) and \( |b_n - b'_n| < \frac{\epsilon}{2B} \) for all \( n \geq M_2 \). This implies that

\[
2|a_n b_n - a'_n b'_n| = |(a_n - a'_n)(b_n + b'_n) + (a_n + a'_n)(b_n - b'_n)|
\leq |(a_n - a'_n)(b_n + b'_n)| + |(a_n + a'_n)(b_n - b'_n)| = |a_n - a'_n| \cdot |b_n + b'_n| + |a_n + a'_n| \cdot |b_n - b'_n|
\leq |a_n - a'_n| \cdot (|b_n| + |b'_n|) + (|a_n| + |a'_n|) \cdot |b_n - b'_n|
\leq \frac{\epsilon}{2B} \cdot (2B) + \frac{\epsilon}{2B} \cdot (2B) = 2\epsilon
\]

for all \( n \geq \max\{N_2, M_2\} \). Thus, \( |a_n b_n - a'_n b'_n| < \epsilon \) for all \( n \geq \max\{N_2, M_2\} \) and \( \{a_n b_n\} \sim \{a'_n b'_n\} \).

This shows that multiplication is well-defined.

**Theorem 1:** The real numbers are a field.

**Proof:** Since the rational real numbers are a field,

\[
([\{a_n\}] + [\{b_n\}]) + [\{c_n\}] = ([\{a_n + b_n\} + c_n]) = ([\{a_n\} + (b_n + c_n)]) = ([\{a_n\}] + ([\{b_n\}] + [\{c_n\}]])
\]

for any Cauchy sequences \( \{a_n\}, \{b_n\} \) and \( \{c_n\} \), and similarly for multiplication. Also,

\[
[\{a_n\}] + [\{b_n\}] = [\{a_n + b_n\}] = [\{b_n + a_n\}] = [\{b_n\}] + [\{a_n\}]
\]

and similarly for multiplication. Next,

\[
([\{a_n\}] + [\{b_n\}]) \cdot [\{c_n\}] = [\{(a_n + b_n) \cdot c_n\}] = [\{(a_n \cdot c_n) + (b_n \cdot c_n)\}] = ([\{a_n\}] \cdot [\{c_n\}]) + ([\{b_n\}] \cdot [\{c_n\}]).
\]
Since 0 is the additive identity element of \( \mathbb{Q} \),
\[
[{}^{0\text{th}}] + [\{a_n\}] = [\{0 + a_n\}] = [\{a_n\}] 
\]
for any \([\{a_n\}] \in \mathbb{R}\), so that \([{}^{0\text{th}}]\) is an additive identity for \( \mathbb{R} \). Clearly
\[
[\{a_n\}] + [\{-a_n\}] = [\{a_n + (-a_n)\}] = [{}^{0\text{th}}]
\]
for any \([\{a_n\}] \in \mathbb{R}\), so that \([\{-a_n\}]\) is an additive inverse for \([\{a_n\}]\).

Since 1 is the multiplicative identity element of \( \mathbb{Q} \),
\[
[\{a_n\}] \cdot [{}^{1\text{st}}] = [\{a_n \cdot 1\}] = [\{a_n\}]
\]
for all \([\{a_n\}] \in \mathbb{R}\). Clearly \( {}^{0\text{th}} \) \( \not\sim {}^{1\text{st}} \). Suppose that \([\{a_n\}] \neq [{}^{0\text{th}}]\), i.e., \( \{a_n\} \neq 0\). This means that for some \( \epsilon > 0 \), there are infinitely many \( n \in \mathbb{N} \) such that \( |a_n| \geq \epsilon \). Since \( \{a_n\} \) is a Cauchy sequence, there is an \( N \in \mathbb{N} \) such that \( |a_n - a_m| < \epsilon / 2 \) for all \( n, m \geq N \). Since there must be an \( n_0 \geq N \) such that \( |a_{n_0}| \geq \epsilon \), then
\[
|a_m| \geq |a_{n_0}| - |a_{n_0} - a_m| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2}
\]
for all \( m \geq N \). In particular, \( a_m \neq 0 \) for \( m \geq N \). Further, it is not difficult to see that the sequence \( \{b_n\} \) defined by
\[
b_n = \begin{cases} 
a_N & \text{if } n \leq N, 
a_n & \text{if } n \geq N 
\end{cases}
\]
is a Cauchy sequence equivalent to \( \{a_n\} \). Thus, we may assume without loss of generality that \( a_n \neq 0 \) for all \( n \in \mathbb{N} \). Now since the rational numbers are a field,
\[
[\{a_n\}] \cdot [\{a_n^{-1}\}] = [\{a_n \cdot a_n^{-1}\}] = [{}^{1\text{st}}],
\]
so that \([\{a_n\}]\) has a multiplicative inverse.

**Definition:** For \([\{a_n\}] \in \mathbb{R}, [\{\{a_n\}\}] = [\{[a_n]\}]\).

**Lemma:** This is well-defined, i.e., \([\{\{a_n\}\}] \in \mathbb{R}\) and depends only on the equivalence class of \( \{a_n\} \).

**Proof:** Let \( \epsilon > 0 \). Then for some \( N \in \mathbb{N} \) we have \( |a_n - a_m| < \epsilon \) for all \( n, m \geq N \). Since \( |a_n| - |a_m| \leq |a_n - a_m| \), this implies that \( |a_n| - |a_m| < \epsilon \) for all \( n, m \geq N \) and \( \{|a_n|\} \) is a Cauchy sequence.
Suppose \( \{a'_n\} \sim \{a_n\} \) and let \( \epsilon > 0 \). Then for some \( N \in \mathbb{N} \), \( |a'_n - a_n| < \epsilon \) for all \( n \geq N \). As above, this implies that \( |a'_n' - a_n| < \epsilon \) for all \( n \geq N \), so that \( \{a'_n\} \sim \{|a_n|\} \).

**Definition:** We say a real number \([\{a_n\}]\) is greater than zero (or positive), and write \([\{a_n\}] > 0\), if there is an \( N \in \mathbb{N} \) and an \( \epsilon > 0 \) such that \( a_n \geq \epsilon \) for all \( n \geq N \).

**Lemma:** This is well-defined.

**Proof:** Suppose \( \{a'_n\} \) and \( \{a_n\} \) are equivalent Cauchy sequences. Suppose further that there is some \( N \in \mathbb{N} \) and an \( \epsilon > 0 \) such that \( a_n \geq \epsilon \) for all \( n \geq N \). There is an \( M \in \mathbb{N} \) such that \( |a'_n - a_n| < \epsilon/2 \) for all \( n \geq M \). This implies that \( |a'_n| \geq |a_n| - a'_n - a_n| > \epsilon - \frac{\epsilon}{2} = \frac{\epsilon}{2} \) for all \( n \geq \max\{N, M\} \).

**Definition:** We say a real number \([\{a_n\}]\) is greater than a real number \([\{b_n\}]\), and write \([\{a_n\}] > [\{b_n\}]\), if \([\{a_n\}] - [\{b_n\}] > 0\).

**Theorem 2:** For any \([\{a_n\}], [\{b_n\}] \in \mathbb{R}\), the following three properties hold:

a) \([|\{a_n\}| \geq |\{a'_n\}|, \text{with equality if and only if } [\{a_n\}] = [\{a'_n\}],\]

b) \([|\{a_n\}| \cdot [\{b_n\}] = [|\{a_n\}| \cdot [\{b_n\}]|];\]

c) \([|\{a_n\}| + [\{b_n\}] \leq [|\{a_n\}|] + [\{b_n\}]|.

In other words, \(| \cdot |\) is an absolute value on \( \mathbb{R}\).

**Proof:** Starting with the first property, we have \(|a_n| \geq 0\) for all \( n \). Suppose that \( \{a_n\} \neq \{0\} \).

Then there must be an \( \epsilon > 0 \) such that \(|a_n| \geq \epsilon\) for infinitely many \( n \in \mathbb{N} \). Let \( N \in \mathbb{N} \) be such that \(|a_n - a_m| < \epsilon/2\) for all \( n, m \geq N \). Since there is an \( n_0 \geq N \) with \(|a_{n_0}| \geq \epsilon\), we have \(|a_m| \geq |a_{n_0}| - |a_{n_0} - a_m| > \epsilon - \epsilon/2 = \epsilon/2\) for all \( m \geq N \). Thus, \([|\{a_n\}|] = [|\{a_n\}|] > [\{0\}].

For the second property, using \(|a_n b_n| = |a_n| \cdot |b_n|\) for all \( n \), we have

\[ [\{a_n\}] \cdot [\{b_n\}] = [|\{a_n b_n\}|] = [|\{a_n\}| \cdot [|\{b_n\}|] = [|\{a_n\}|] \cdot [\{b_n\}]. \]

Now for the third property - the “triangle inequality.” Suppose first that there is some \( \epsilon > 0 \) and some \( N \in \mathbb{N} \) such that \(|a_n + b_n| + \epsilon \leq |a_n| + |b_n|\) for all \( n \geq N \). Then by the definitions,

\[ [|\{a_n\}| + [\{b_n\}] = [|\{a_n + b_n\}|] < [|\{a_n\}| + [\{b_n\}]| = [|\{a_n\}|] + [\{b_n\}]. \]

So suppose this is not the case and let \( \epsilon > 0 \). Then there are infinitely many \( n \in \mathbb{N} \) such that \(|a_n + b_n| + \epsilon/2 > |a_n| + |b_n|\). Also, there is an \( N \in \mathbb{N} \) such that

\[ |a_n - a_m|, |b_n - b_m|, |a_n + b_n| - |a_m + b_m| < \epsilon/6 \]
for all \( n, m \geq N \). Choose an \( n \geq N \) such that \(|a_n + b_n| + \epsilon/2 > |a_n| + |b_n|\). Then for all \( m \geq N \) we have

\[
|a_m| + |b_m| - |a_m + b_m| < |a_n| + \frac{\epsilon}{6} + |b_n| + \frac{\epsilon}{6} - |a_n + b_n| + \frac{\epsilon}{6}
\]

\[
= |a_n| + |b_n| - |a_n + b_n| + \frac{\epsilon}{2}
\]

\[
< \epsilon.
\]

By the triangle inequality for rational numbers, \(|a_m| + |b_m| \geq |a_m + b_m|\). Thus,

\[
||a_m| + |b_m| - |a_m + b_m|| < \epsilon
\]

for all \( m \geq N \) and \(|a_n + b_n|\sim|a_n| + |b_n|\). By the definitions, this means that

\[
|\{(a_n)\} + \{(b_n)\}| = |\{(a_n)\}| + |\{(b_n)\}|.
\]

**Lemma:** Suppose \(|\{(a_n)\}|\) and \(|\{(b_n)\}|\) are two unequal real numbers. Then there is a rational number \(c\) such that \(|\{(a_n)\} - \{\text{“}c\text{”}\}| < |\{(a_n)\}| - |\{(b_n)\}|\).

**Proof:** Since \(a_n \neq \{b_n\}\), \(a_n - b_n\) \text{ “}0\text{”}. By Theorem 2, \(|\{(a_n)\} - \{(b_n)\}| = |\{(a_n - b_n)\}| > 0\) so that there is an \( \epsilon > 0 \) and an \( N \in \mathbb{N} \) such that \(|a_n - b_n| \geq \epsilon\) for all \( n \geq N \). Also, for some \( M \in \mathbb{N} \) we have \(|a_n - a_m| < \epsilon/2\) for all \( n, m \geq M \). Let \( c = a_{N + M} \). Then for all \( n \geq N + M \), \(|a_n - c| < \epsilon/2\) and \(|a_n - b_n| \geq \epsilon\). In particular, \(|a_n - b_n| - |a_n - c| \geq \epsilon/2\) for all \( n \geq N + M \). By the definitions, this means that \(|\{(a_n)\} - \{\text{“}c\text{”}\}| < |\{(a_n)\}| - |\{(b_n)\}|\).

**Theorem 3:** Every Cauchy sequence of real numbers converges, i.e., the real numbers are a topologically complete field.

[Technically speaking, the \( \epsilon \) in the definition of Cauchy sequence of real numbers is allowed to be real. However, it suffices to restrict to the case where \( \epsilon \) is a positive rational number by the lemma above.]

**Proof:** Let \( \{r_n\}\) be a Cauchy sequence of real numbers. If there is an \( r \in \mathbb{R} \) and an \( N \in \mathbb{N} \) such that \( r_n = r \) for all \( n \geq N \) we are done, since in this case \( \{r_n\} \to r \). So suppose this is not the case. For each \( n \in \mathbb{N} \) let \( n' \in \mathbb{N} \) be least such that \( n' > n \) and \( r_n \neq r_{n'} \). For each \( n \in \mathbb{N} \) choose (via the lemma above) an \( a_n \in \mathbb{Q} \) such that \(|r_n - \{\text{“}a_n\text{”}\}| < |r_n - r_{n'}|\).

Let \( \epsilon > 0 \). Then there is an \( N \in \mathbb{N} \) such that \(|r_n - r_m| < \epsilon/3\) for all \( n, m \geq N \). By the triangle
implies that the sum of two positive real numbers. Suppose $x < y$ and $z > x$. Thus, $z > x$.

**Theorem 4:** The real numbers are totally ordered by $\lt$. Suppose $x < y$. By definition, this means $y - x > 0$. Since $(y + z) - (x + z) = y - x$, we have $y + z > x + z$. Suppose $x < y$ and $y < z$. Then $z - x = (z - y) + (y - x)$ is positive, being the sum of two positive real numbers. Suppose $x < y$ and $z > 0$. Then $z(y - x)$ is positive, so that $zy > zx$.

Suppose $x \not< y$ and $y \not< x$, and write $x = \{a_n\}, y = \{b_n\}$. Let $\epsilon > 0$. There are infinitely many $n \in \mathbb{N}$ such that $a_n - b_n < \epsilon/3$ and infinitely many $n \in \mathbb{N}$ such that $b_n - a_n < \epsilon/3$. Also, there are $N, M \in \mathbb{N}$ such that $|a_n - a_m| < \epsilon/6$ for all $n, m \geq N$ and $|b_n - b_m| < \epsilon/6$ for all $n, m \geq M$. For some $m$ and $n$, we have

$$|a_n - a_m| = \left|\left\lceil \frac{a_n}{\epsilon} \right\rceil - \left\lceil \frac{a_m}{\epsilon} \right\rceil\right| = \left|\frac{a_n}{\epsilon} - \frac{a_m}{\epsilon} + \frac{a_m}{\epsilon} - \frac{a_m}{\epsilon} - \left\lceil \frac{a_m}{\epsilon} \right\rceil\right|$$

for all $n, m \geq N$. This shows that $\{a_n\}$ is a Cauchy sequence (of rational numbers), i.e., $\{a_n\} \in \mathbb{R}$.

Let $\epsilon > 0$ again. Then there are is an $N \in \mathbb{N}$ such that $|r_n - r_m|, |a_n - a_m| < \epsilon/3$ for all $n, m \geq N$. In particular, $|r_N - \left\lceil a_N \right\rceil| < \epsilon/3$ and also $|\left\lceil a_N \right\rceil - |\{a_m\}|| < \epsilon/3$. Using the triangle inequality once more,

$$|r_n - \{a_n\}| = |r_n - r_N + r_N - \left\lceil a_N \right\rceil + \left\lceil a_N \right\rceil - |\{a_m\}|$$

$$\leq |r_n - r_N| + |r_N - \left\lceil a_N \right\rceil| + |\left\lceil a_N \right\rceil - |\{a_m\}||$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$$

$$= \epsilon$$

for all $n \geq N$. Thus, $\{r_n\} \to \{\{a_m\}\} \in \mathbb{R}$.

**Lemma:** The sum and product of two positive real numbers is positive.

**Proof:** Suppose $\{\{a_n\}\}$ and $\{\{b_n\}\}$ are positive real numbers. Then by definition there are $N_1, N_2 \in \mathbb{N}$ and $\epsilon_1, \epsilon_2 > 0$ such that $a_n \geq \epsilon_1$ for all $n \geq N_1$ and $b_n \geq \epsilon_2$ for all $n \geq N_2$. This implies that $a_n + b_n \geq \epsilon_1 + \epsilon_2 > 0$ and $a_n b_n \geq \epsilon_1 \epsilon_2 > 0$ for all $n \geq \max\{N_1, N_2\}$. In other words, $\{a_n\} + \{b_n\} > 0$ and $\{a_n\} \cdot \{b_n\} = \{a_n b_n\} > 0$.


Thus \(n, m \geq M\). Choose an \(n_0, m_0 \geq \max\{N, M\}\) such that \(a_{n_0} - b_{n_0} < \varepsilon / 3\) and \(b_{m_0} - a_{m_0} < \varepsilon / 3\). Then
\[b_{n_0} - a_{n_0} < b_{m_0} + \varepsilon / 6 - a_{m_0} + \varepsilon / 6 < 2\varepsilon / 3.\]
For any \(n \geq \max\{N, M\}\) we have
\[
|a_n - b_n| = |a_n - a_{n_0} + a_{n_0} - b_{n_0} + b_{n_0} - b_n|
\leq |a_n - a_{n_0}| + |a_{n_0} - b_{n_0}| + |b_{n_0} - b_n|
< \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} + \frac{\varepsilon}{6}
= \varepsilon.
\]
Thus \(\{a_n - b_n\} \to 0\) and \([\{a_n\}] = [\{b_n\}]\). This shows that either \(x < y\) or \(y < x\) or \(x = y\).

Finally, suppose \(x > y\) and \(y < x\). Then \([“0”] = (x - y) + (y - x)\) is positive since it is the sum of two positive numbers. Similarly, if \(x = y\) and either \(x > y\) or \(y > x\), then \([“0”]\) is positive, which is clearly not the case, so that at most one of \(x > y\), \(y > x\) and \(x = y\) holds.

**Theorem 5:** Suppose \(S \subset \mathbb{R}, S \neq \emptyset\), and there is a \(B \in \mathbb{R}\) with \(x \leq B\) for all \(x \in S\). (We say \(B\) is an upper bound for \(S\).) Then there is an upper bound \(b \in \mathbb{R}\) for \(S\) such that for all \(c < b\) there is an \(x \in S\) with \(x > c\). (We say \(b\) is a least upper bound for \(S\).)

**Proof:** Chose an \(x_0 \in S\) and let \(m_0 \in \mathbb{N}\) be largest such that \(B - m_0\) is an upper bound for \(S\). Then \(m_0\) exists since \(B - 0\) is an upper bound for \(S\) and \(B - n\) is not an upper bound for \(S\) whenever \(n > B - x_0\).

We claim that there is a sequence \(B_1, B_2, \ldots\) of upper bounds for \(S\) such that \(B_n - 2^{-n}\) is not an upper bound for \(S\) and \(0 \leq B_{n-1} - B_n \leq 2^{-n}\) for all \(n \in \mathbb{N}\). We prove this claim by induction.

Let \(m_1 \in \mathbb{N}\) be largest such that \(B_0 - m_1 2^{-1}\) is an upper bound for \(S\). Then \(m_1\) is either 0 or 1 since \(B_0 - 0\) is an upper bound for \(S\) and \(B_0 - 1 = B_0 - 2 \cdot 2^{-1}\) is not. Let \(B_1 = B_0 - m_1 2^{-1}\). Then \(B_1\) is an upper bound for \(S\) and \(0 \leq m_1 2^{-1} = B_0 - B_1 \leq 2^{-1}\). Also, \(B_1 - 2^{-1} = B_0 - (m_1 + 1)2^{-1}\) in not an upper bound for \(S\) by the definition of \(m_1\).

Now suppose \(B_1, \ldots, B_n\) have been chosen which satisfy the requirements above. Let \(m_{n+1}\) be the largest integer such that \(B_n - m_{n+1} 2^{-n-1}\) is an upper bound for \(S\). By the induction hypothesis, \(B_n\) is an upper bound for \(S\) and \(B_n - 2^{-n}\) is not, so that \(m_{n+1}\) is either 0 or 1. Let \(B_{n+1} = B_n - m_{n+1} 2^{-n-1}\). Then \(B_{n+1}\) is an upper bound for \(S\) and \(0 \leq m_{n+1} 2^{-n-1} = B_n - B_{n+1} \leq 2^{-n-1}\). Finally, \(B_{n+1} - 2^{-n-1} = B_n - (m_{n+1} + 1)2^{-n-1}\) is not an upper bound for \(S\) by the definition of \(m_{n+1}\).

We claim that this is a Cauchy sequence. Indeed, let \(\varepsilon > 0\) and suppose \(n, m \in \mathbb{N}\) with \(n > m\).
Then
\[ B_n - B_m = (B_n - B_{n-1}) + \cdots + (B_{m+1} - B_m) \]
\[ \leq 2^{-n} + \cdots + 2^{-m-1} \]
\[ = 2^{-m-1}(1 + \cdots + 2^{m+1-n}) \]
\[ = 2^{-m-1} \frac{1 - 2^{m-n}}{1 - 2^{-1}} \]
\[ = 2^{-m}(1 - 2^{m-n}) \]
\[ < 2^{-m}. \]

Thus, if \( n, m \geq N \) where \( 2^{-N} \leq \epsilon \), then \( |B_n - B_m| < \epsilon \).

Let \( b \in \mathbb{R} \) be the limit of this Cauchy sequence. Let \( x \in S \). Suppose \( x > b \) and let \( \epsilon = x - b \). For some \( n \in \mathbb{N} \), \( |B_n - b| < \epsilon \). But this implies that \( x - b > |B_n - b| \), so that \( x > B_n \). This contradicts the fact that all \( B_n \)'s are upper bounds for \( S \). Thus, \( b \) is an upper bound for \( S \).

Finally, suppose \( c < b \) is an upper bound for \( S \). Then \( b - c \geq 2^{-n_0} \) for some \( n_0 \in \mathbb{N} \), which implies that \( b - 2^{-n_0} \) is an upper bound for \( S \). Choose \( N \in \mathbb{N} \) such that \( |B_n - b| < 2^{-n_0-1} \) for all \( n \geq N \). We then have \( B_n - 2^{-n_0-1} > b - 2^{-n_0} \) for all \( n \geq N \). But \( B_n - 2^{-n} \) is not an upper bound for any \( n \), so that \( B_n - 2^{-n_0-1} \leq B_n - 2^{-n} \) is not an upper bound for any \( n > n_0 \). This contradiction shows that \( c \geq b \), so that \( b \) is a least upper bound for \( S \).