Shift-Invariant Spaces in the Fractional Fourier Transform Domain

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Outline

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Shift-Invariant Spaces in the Fractional Fourier Transform Domain

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The fractional Fourier transform gained very much popularity in the early 1990s because of its numerous applications in signal analysis and optics.

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For light propagation in quadratic graded-index media (fiber optics), it is known that the Fourier transform is produced at a certain distance $d_0$ that depends on the medium. Thus, it is reasonable to call the light distribution at distance $ad_0$, $0 < a \leq 1$, the fractional Fourier transform of order $a$. 

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Shift-Invariant Spaces in the Fractional Fourier Transform Domain
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$$W_f(u, v) = \int_{\mathbb{R}} f(u + x/2) f^*(-u - x/2) e^{-2\pi ivx} dx.$$
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It is related to the Radar ambiguity function. The Wigner distribution of $\hat{W_f}(u, v)$ is obtained from $W_f(u, v)$ by a rotation of $\pi/2$.

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What does correspond to a rotation by an angle $\pi/4$? Whatever it is, we call it the one half Fourier transform.
More generally, what does correspond to a rotation by an angle $\theta$? i.e., Find $g$ such that

$$W_g(u, v) = W_f(u \cos \theta - v \sin \theta, u \sin \theta + v \cos \theta).$$
More generally, what does correspond to a rotation by an angle \( \theta \)? i.e., Find \( g \) such that

\[
W_g(u, v) = W_f(u \cos \theta - v \sin \theta, u \sin \theta + v \cos \theta).
\]

\( g \) is the fractional Fourier transform with angle \( \theta \).
The Fractional Fourier transform may also be viewed as a family of bounded operators $\mathcal{F}_\alpha$, with $0 \leq \alpha \leq 1$, such that

$$\mathcal{F}_0(f) = f, \quad \mathcal{F}_1 = \hat{f}.$$
The Fractional Fourier transform may also be viewed as a family of bounded operators $\mathcal{F}_\alpha$, with $0 \leq \alpha \leq 1$, such that

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In practice, it is indexed by an angle $0 \leq \theta \leq 2\pi$ so that

$$\mathcal{F}_0(f) = f, \quad \mathcal{F}_{\pi/2} = \hat{f}, \quad \mathcal{F}_\pi (f(x)) = f(-x), \quad \mathcal{F}_{2\pi} = f.$$
The fractional Fourier Transform (FrFT)

\[ \hat{x}_\theta(\omega) = \int x(t) e^{j\frac{\omega^2 + t^2}{2} \cot \theta - j\omega t \csc \theta} dt \]

**Properties:**
- Bandlimitedness
- Orthonormality
- Preserves L₂–norm (Parseval – Plancheral)
- Hermite polynomials are Eigen Functions
- Semi-group Property
Fractional Form of Fourier Operator – I

\[ \mathbf{FT} \{ \Psi_n(t) \} = \langle \Psi_n(t), e^{j\omega t} \rangle = \lambda_n \Psi_n(\omega) \]

Normalized Hermite Polynomials

\[ \Psi_n(t) = \frac{2^{1/4}}{\sqrt{2^n n!}} H_n(\sqrt{2\pi t}) \exp \left( -\pi t^2 \right) \]

Eigen Functions

\[ \langle \Psi_n, \Psi_m \rangle = \delta_{m,n} \]

Eigen Values

\[ \lambda_n = e^{jn\pi/2} \]
Possible because Eigen-functions are Orthonormal!

$$\left\{ \underbrace{\text{FT} \cdots \text{FT}}_{p\text{-times}} \{\Psi_n(t)\} \right\} = \langle \Psi_n(t), e^{j\omega t} \rangle = \lambda_n^p \Psi_n(\omega)$$

$$\text{FT}^p \{f(t)\} = \sum_{n=-\infty}^{\infty} \langle f, \Psi_n \rangle \cdot \lambda_n^p \Psi_n(\omega)$$

$$K_p(t, \omega) = \sum_{n=-\infty}^{\infty} \lambda_n^p \cdot \Psi_n^*(t) \Psi_n(\omega)$$

Key words: Mercer’s Formula, Reproducing Kernel Hilbert Space...
Sampling of Sparse Signals in FrFT Domain

Signal of Interest: Sparse in Time

\[ x(t) = \sum_{k=0}^{K-1} \sum_{n} c_k \delta(t - t_k - n\tau) \]

Periodic stream of Diracs with TWO K Degrees of Freedom

Innovations: Time instances \( \{t_k\} \) and Amplitudes \( \{c_k\} \)

Rate of innovation: Number of degrees of freedom per unit of time.
Shannon’s Sampling Theorem for FrFT Domain

**Theorem** (Shannon FrFT: Zayed-96, Xia-96, Erseghe-99, Garcia-00, Candan-03, Torres-06, Tao-08). Let $x(t)$ be a continuous-time signal. If the spectrum of $x(t)$, i.e. $\hat{x}_\theta(\omega)$ is fractional bandlimited to $\Omega_m$, then $x(t)$ is completely determined by giving its ordinates at a series of equidistant points spaced $T = \frac{\pi}{\Omega_m} \sin \theta$ seconds apart. And thus,

$$x(t) = e^{-j\frac{\cot \theta}{2} t^2} \sum_{n=-\infty}^{+\infty} x(nT)e^{j\frac{\cot \theta}{2} (nT)^2} \text{sinc} \left((t - nT)\omega_m \csc \theta\right)$$

*Extension of Shannon’s sampling theorem to FrFT shows its association to a Nyquist-like bound.*
Quick Proof! (Sampling: 50 Years After Shannon ...)

\[ \varphi_n(t) = e^{-j \frac{t^2}{2}} \cot \theta \text{sinc}(t - nT) \]

For such an orthonormal family of functions, i.e. \( \{\varphi_n(t)\}_{n=-\infty}^{\infty} \), we know that \( x \in V \iff x = \mathcal{P}_V x \), where \( V \) is the approximation, so as,

\[ V(\varphi) = \left\{ x(t) = \sum_{n \in \mathbb{Z}} \langle x, \varphi_n \rangle \varphi_n(t) : \langle x, \varphi_n \rangle \in l_2 \right\}. \]

\[ \langle x, \varphi_n \rangle = x(nT) e^{j \frac{\cot \theta}{2} (nT)^2} \]

\[ \mathcal{P}_V x = e^{-j \frac{\cot \theta}{2} t^2} \sum_{n \in \mathbb{Z}} x(nT) e^{j \frac{\cot \theta}{2} (nT)^2} \text{sinc}(t - nT) \]
Sampling of Sparse Signals in FrFT Domain

Signal of Interest: Sparse in Time

\[ x(t) = \sum_{m} \hat{x}[m] \Phi_{\pi/2}(m, t) \]

**Expansion in Fourier Basis**

\[ \Phi_{\pi/2}(m, t) \]

\[ \text{generalization} \]

\[ x(t) = \sum_{m} \hat{x}_\theta[m] \Phi(m, t) \]

**Expansion in Fractional Fourier Basis**

The Poisson summation formula!

\[ \Phi^*_\theta(m, t) = \sqrt{\frac{\sin \theta - j \cos \theta}{T}} e^{\frac{j t^2 + (2\pi m \sin \theta / \tau)^3}{2} \cot \theta - \frac{j 2\pi m t}{\tau}} \]
Sampling of Sparse Signals in FrFT Domain

\[
\sum_{\ell \in \mathbb{Z}} \delta(t - \ell T) = \sqrt{\frac{1}{T}} \sum_{k \in \mathbb{Z}} \tilde{\delta}_\theta (k \omega_0 \sin \theta) \cdot e^{-j \left( \frac{t^2}{2} + \frac{k \omega_0 \sin \theta}{\tau} \right)} e^{j k \omega_0 t}.
\]

The Poisson summation formula for Fractional Fourier Transform Domain

\[
x(t) = e^{\frac{-jt^2 \cot \theta}{2}} \sum_{m \in \mathbb{Z}} \left( \sum_{k=0}^{K-1} c_k e^{\frac{j \cot \theta}{2} \left( i_k^2 - jmk_0 \right)} \right) e^{\frac{j 2\pi m}{\tau} t}
\]

Bon! The knowledge of K-complex exponentials is good enough for sampling signals in FrFT Domain.

\[
x(t) = \sum_{m \in \mathbb{Z}} \frac{1}{\tau} \left( \sum_{k=0}^{K-1} c_k e^{-j(2\pi mt_k / \tau)} \right) e^{j(2\pi mt / \tau)}
\]

Fourier Series Coefficients of some expansion
Theorem (FrFT – FRI): Let $x(t)$ be a $\tau$-periodic stream of Diracs weighted by coefficients $\{c_k\}_{k=0}^{K-1}$ and locations $\{t_k\}_{k=0}^{K-1}$ with finite rate of innovation $\rho = \frac{2K}{\tau}$. Let the sampling kernel/prefilter $\varphi(t)$ be an ideal low-pass filter which has fractional bandwidth $[-B\pi, B\pi]$, where $B$ is chosen such that $B \geq \rho$. If the filtered version of $x(t)$, i.e. $y(t) = x(t) *_{\theta} \varphi(-t)$ is sampled uniformly at locations $t = nT$, $n = 0, \ldots, N - 1$ then the samples,

$$y(nT) = x(t) *_{\theta} \varphi(-t)|_{t=nT}, n = 0, \ldots, N - 1,$$

are a sufficient characterization of $x(t)$, provided that $N \geq 2M_\theta + 1$ and $M_\theta = \left\lfloor \frac{B\tau \csc \theta}{2} \right\rfloor$. 
Any Function in \( V(\phi) \) can be viewed as a convolution of a sequences \( \{ c(k) \} \in \ell^2 \) and a function \( \phi \in L^2(\mathbb{R}) \), where the convolution is defined as

\[
(c(k) \ast \phi)(t) = \sum_{k \in \mathbb{Z}} c(k) \phi(t - k).
\]
Going over to the fractional Fourier domain, we let
\[ \lambda_\theta (t) = \exp \left( j \left( \frac{t^2}{2} \right) \cot \theta \right) \]
be a modulation function.
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\( \lambda_\theta (t) = \exp (j (t^2/2) \cot \theta) \) be a modulation function. The chirp modulated and demodulated versions of a signal \( x(t) \) are respectively defined by

Modulation/up-chirping: \( \tilde{x}(t) = x(t)\lambda_\theta(t) \)

Demodulation/down-chirping: \( \hat{x}(t) = x(t)\lambda^*_\theta(t) \).
The fractional convolution of two input signals, $x(t)$ and $y(t)$ is defined as (A. Zayed, IEEE Sign. Proc. Letters, Vol. 5 (1998))

$$x(t) *_{\theta} y(t) = \sqrt{\frac{1 - j \cot \theta}{2\pi}} \lambda^*_\theta(t) \cdot \left( [x(t) \lambda_\theta(t)] * [y(t) \lambda_\theta(t)] \right)$$

convolution of modulated inputs

$$= c(\theta) \lambda^*_\theta(t) \left\{ \hat{x}(t) * \hat{y}(t) \right\}, \quad (1)$$

where $c(\theta) = \sqrt{(1 - j \cot \theta)/2\pi}$.

Which leads to FrFT \{ $x(t) *_{\theta} y(t)$ \} = $\lambda^*_\omega \cdot \hat{x}_\theta(\omega) \hat{y}_\theta(\omega)$. 

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Let $t_k = k\Delta$, where $\Delta = 2\pi \sin \theta$, be a sequence of uniformly spaced real numbers.
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Define the \textbf{discrete Fractional Fourier transform} of a sequence \( \{ x(k) \} \)

\[
\hat{X}_\theta(w) = \sum_{k=-\infty}^{\infty} x(k) K_\theta(k, w)
\]  

(2)
Let \( t_k = k\Delta \), where \( \Delta = 2\pi \sin \theta \), be a sequence of uniformly spaced real numbers. Define the discrete Fractional Fourier transform of a sequence \( \{x(k)\} \)

\[
\hat{X}_\theta(w) = \sum_{k=-\infty}^{\infty} x(k) K_\theta(k, w)
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(2)

and define the convolution of a sequence and a function \( \{x(k)\} \) with \( \phi \in L^2(R) \) as

\[
h(t) = (x(k) *'_\theta \phi)(t) = c(\theta) \bar{\lambda}_\theta(t) \sum_{k=-\infty}^{\infty} \hat{x}(k) \hat{\phi}(t - t_k)
\]
Consider

\[ \tilde{h}(t) = \lambda_\theta(t)h(t) = c(\theta) \sum_{k=\infty}^{\infty} \tilde{x}(k)\tilde{\phi}(t - t_k) \]

where \( \tilde{x}(k) = x(k)jat^2_k, \tilde{\phi}(t) = \phi(t)e^{jat^2}, \) and \( a = \cot \theta. \)

The fractional spectrum of \( h(t) \) is \( \hat{h}_\theta(\omega) = \lambda^*_{\theta}(\omega) \hat{P}_\theta(\omega) \hat{\phi}_\theta(\omega), \)

where \( \hat{P}_\theta \) is the discrete time fractional Fourier transform (DTFrFT) of the sequence \( \{ p(n) = e^{jat^2_n}x(n) \}. \)
Let $\{x(n)\} \in \ell_2$, $\phi \in L^2(R)$ and set

$$
\psi(t) = e^{jat^2} \phi(t), \quad p(n) = e^{jat^2_n} x(n)
$$

and consider the chirp-modulated shift-invariant subspaces of $L^2$

$$
V(\psi) = \text{cl} \left\{ \tilde{f} \in L^2 : \tilde{f}(t) = c(\theta) \sum_{k=-\infty}^{\infty} p(k) \psi(t - t_k) \right\}
$$

and

$$
V(\phi) = \text{cl} \left\{ f \in L^2 : f(t) = (x(n) \ast'_\theta \phi)(t) = \lambda^*_\theta(t) \tilde{f}(t), \ \tilde{f} \in V(\psi) \right\}
$$
Then \( \{\psi(t - t_k)\} \) is a Riesz basis for \( V(\psi) \) if and only if there exist two positive constants \( \eta_1, \eta_2 > 0 \) such that

\[
\eta_1 \leq \sum \left| \hat{\phi}_\theta(w + t_k) \right|^2 \leq \eta_2
\]

for all \( w \in [0, \Delta] \).

Other properties of shift-invariant spaces, such as sampling subspaces, etc, have been obtained in the fractional Fourier domain.
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Thanks for listening.