INSTRUCTIONS:
(1) Print your name and student number (ZID) above.
(2) Make certain that your test has all five (5) different sheets (including the cover page).
(3) You must SHOW YOUR WORK in order to get credit.
(4) Clearly indicate your answer to each problem BY CIRCLING IT.
(5) The point value of each problem is given in the parentheses at the right of the problem number. There are 100 total points possible.
(6) Calculators are not permitted on this exam.
(7) Individual exams will not be curved. All curving will be done when the final grade is determined.
(1) (5 pts.) Mark only linear equations in the list below. You do not need to show any work here.

(a) \( \left( \frac{y}{\sqrt{x}} + 2\sqrt{y} + 2x \right) + \left( \frac{x}{\sqrt{y}} + 2\sqrt{x} + 2 \right) \frac{dy}{dx} = 0 \) – non-linear;
(b) \( e^x y' - y'' = \tan x \) – linear;
(c) \( e^x y' - y'' = \tan y \) – non-linear;
(d) \( xdy - ydx = 0 \) – linear;
(e) \( (y')^2 = 0 \) – non-linear.

(2) (5 pts.) Mark only separable equations in the list below. You do not need to show any work here.

(a) \( \left( \frac{y}{\sqrt{x}} + 2\sqrt{y} + 2x \right) + \left( \frac{x}{\sqrt{y}} + 2\sqrt{x} + 2 \right) \frac{dy}{dx} = 0 \) – non-separable;
(b) \( e^x y' - y = \tan x \) – non-separable;
(c) \( e^x y' - y = \tan y \) – separable;
(d) \( xdy - ydx = 0 \) – separable;
(e) \( (x^2 + y^2)y' = xy \) – non-separable.

(3) (10 pts.) Determine if the following IVP is guaranteed to have a unique solution in some interval by the existence and uniqueness theorem for first order DE. Show your work!

(a) \( y' = 2\sqrt{y}; \quad y(0) = 0. \)

The function \( f(x, y) = 2\sqrt{y} \) is not defined AROUND \((0, 0)\). Therefore, the theorem is not applicable. In fact, a solution exists but is not unique.

(b) \( y' = 2\sqrt{y}; \quad y(0) = 1. \)

Both the function \( f(x, y) = 2\sqrt{y} \) and its partial derivative \( f_y(x, y) = y^{-\frac{1}{2}} \) are continuous around \((0, 1)\). Therefore, the solution exists and is unique.
(4) (20 pts.) Determine the solution of the initial-value problem
\[ yy' = x(y^2 - 1), \quad y(0) = -1 \]

The function \( y(x) = -1 \) obviously solves the IVP. Since both \( f(x, y) = \frac{x}{y}(y^2 - 1) \) and its partial \( f_y(x, y) = x(1 + y^{-2}) \) are both continuous around \((0, 1)\), the solution is unique.

The equation is separable. Those of you who found a one-parameter family of solutions got partial credit.

(5) (20 pts.) Find the general solution of the homogeneous equation
\[ (y^2 - x^2)dx + xydy = 0. \]

The equation is homogeneous. We use the substitution \( y = vx \). We have \( dy = xdv + vdx \). Plugging in:

\[ x^2(v^2 - 1)dx + x^2v(xdv + vdx) = 0; \]
\[ (2v^2 - 1)dx + xvdv = 0 \quad \text{or} \quad x = 0; \]
\[ \frac{vdv}{1 - 2v^2} = \frac{dx}{x} \quad \text{or} \quad x = 0, \quad \text{or} \quad v^2 = 1/2; \]
\[ -\frac{1}{4} \ln|2v^2 - 1| = \ln |x + C| \quad \text{or} \quad x = 0, \quad \text{or} \quad v^2 = 1/2; \]
\[ |2v^2 - 1|^{-\frac{1}{4}} = C|x|, \quad C > 0, \quad \text{or} \quad x = 0, \quad \text{or} \quad v^2 = 1/2; \]
\[ |2v^2 - 1|^{\frac{1}{4}}x = C, \quad C \in \mathbb{R}. \]

Finally, substituting back and raising both sides to the 4th power, we get
\[ 2x^2y^2 - x^4 = C, \quad C \in \mathbb{R}. \]

Observe that all the “equilibrium” solutions are incorporated into this formula. You may also notice that the function
\[ F(x, y) = 2x^2y^2 - x^4 \]
can be obtained if one solves the exact equation
\[ 4x(y^2 - x^2)dx + 4x \cdot xydy = 0, \]
which is, of course, equivalent to the one we solved.
(6) (20 pts.) Consider the differential equation
\[(\sin x \cdot \sin y) dy - (\cos x \cdot \cos y) dx = 0 .\]

(a) Show that this is an exact differential equation.

\[M(x, y) = -\cos x \cdot \cos y; \quad N(x, y) = \sin x \cdot \sin y.\]
\[M_y(x, y) = \cos x \cdot \sin y = N_x(x, y).\]

Hence, the equation is exact.

(b) Determine the implicit general solution of this differential equation.

\[F(x, y) = \int M(x, y) dx + g(y) = \]
\[= - \int \cos x \cdot \cos y dx + g(y) = - \sin x \cdot \cos y + g(y).\]
\[F_y(x, y) = \sin x \cdot \sin y + g'(y) = \sin x \cdot \sin y = N(x, y).\]

Hence, the equation is exact.

\[g = 0 \text{ and } \sin x \cdot \cos y = C, \quad C \in \mathbb{R},\]

is the general solution.
(7) (20 pts.) A certain drug is being administered intravenously to a hospital patient. Fluid containing 5 mg/cm$^3$ of the drug enters the patient’s bloodstream at a rate of 100 cm$^3$/hr. The drug is absorbed by body tissues (leaves the bloodstream) at a rate of 0.4 mg/hr per mg of the amount present. Let $Q$ represent the quantity of the drug in the bloodstream, measured in mg, and let $t$ represent time, measured in hours. Write a linear differential equation involving $Q$ and $t$, modeling the system. Find the general solution of the equation. What is the amount of drug in the bloodstream after 1 hour of the treatment, assuming that there was none when it started? SHOW ALL YOUR WORK.

Note: this is not quite the mixture problems we had in class. However, it is sufficiently similar to those (in fact, simpler), which makes me believe that you are capable of doing it. You will get partial credit for solving an equation you would write even if it is not the correct one.

\[
\frac{dQ}{dt} = 5 \cdot 100 - 0.4Q = 500 - 0.4Q.
\]

\[
\frac{dQ}{dt} + 0.4Q = 500.
\]

$p(t) = 0.4$, $q(t) = 500$. The integrating factor

\[
m(t) = e^{\int 0.4 dt} = e^{0.4t}
\]

\[
e^{0.4t}Q = \int 500e^{0.4t} dt = 1250e^{0.4t} + C.
\]

\[
Q(t) = Ce^{-0.4t} + 1250
\]

is the general solution of the equation.

Since $Q(0) = 0$, we get $C = -1250$, $Q(t) = 1250(1 - e^{-0.4t})$, and $Q(1) = 1250(1 - e^{-0.4})$ mg.