Spatial Relationships

Area

**Question:** How big is an acre?

Which is bigger: an acre or a football field?

Webster Unabridged Dictionary: An acre is a U. S. and English measurement, meant to represent the amount of land a yoke of oxen could plow in one day.

One acre equals 160 square rods

What’s a rod? By Webster again,

One rod equals 5.5 yards

1 square rod = $5.5^2$ square yards = 30.25 square yards

1 acre = $160 \times 30.25$ square yards = 4840 square yards

Now a collegiate football field measures 100 by $53\frac{1}{3}$ yards.

a football field = 5333 $\frac{1}{3}$ square yards

Thus we see

a football field is about 10% bigger than an acre.

**Example 1.** III.1, page 166, #23.

**Example 2.** Shadows. Old Egyptian Probem: How do you measure the height of the very top of a pyramid?
1. Plant a stick so that exactly 6 feet stick straight up, perpendicular to the ground.

2. When the shadow is exactly 6 feet long (position Y), mark the tip of the shadow of the pyramid (position B).

3. When the shadow is another 6 feet long from Y (position Z), mark the tip of the shadow of the pyramid (position C).

The height of the pyramid is the length from B to C.

**Example 3.** Shadows. How high is the top of a telephone pole in front of your house?

Compare page 192, Problem 2.
1. Plant a stick so that exactly 4 feet stick straight up, perpendicular to the ground.

2. At some point in the day mark the tip of the shadow of the pole (position Y).

3. At exactly the same time, have your friend mark the tip of the shadow of the tree (position C).

By similar triangles,

\[ \frac{AB}{BC} = \frac{XY}{YZ} \]

where

- \( AB \) = the height of the tree
- \( BC \) = the length of the shadow of the tree
- \( XY \) = the height of the pole = 4
- \( YZ \) = the length of the shadow of the pole

If the shadow of the pole is 2 feet long, and the shadow of the tree is 20 feet long, then our equation becomes

\[ \frac{AB}{20} = \frac{4}{2} \]
or

$$AB = \frac{4}{2} \cdot 20 = 40$$

so the tree is 40 feet tall.

(If the shadow of the tree is 10 times the shadow of the pole, then the height of the tree is 10 times the height of the pole.)

The Pythagorean Theorem

Some definitions:

An angle of 90 degrees is called a **right angle**.

A **right triangle** is a triangle where one angle is a right angle.

The side opposite the right angle is called the **hypotenuse**.

The other two sides are called the **legs** of the right triangle.

Area of a rectangle whose sides measure $a$ and $b$:

$$\text{Area } A = a \times b.$$ 

Area of a general triangle:

$$\text{Area } A = \frac{1}{2} \text{ base } \times \text{ height}.$$ 

Area of a general triangle whose legs measure $a$ and $b$:

$$\text{Area } A = \frac{1}{2} a \times b.$$ 

The Pythagorean Theorem: If the legs of a right triangle measure $a$ and $b$, and the hypotenuse has length $c$, then

$$a^2 + b^2 = c^2$$

Common Examples:

$$3 - 4 - 5$$

$$5 - 12 - 13$$
Example 4. The gate of the Challand measures 5 feet high by 10 feet long. If a wire runs diagonally from the lower left hand corner to the upper right hand corner, how long is the wire?

Compare page 193, Problem 10.

Solution: Use the Pythagorean Theorem with $a = 5$ and $b = 10$:

$$c^2 = a^2 + b^2 = 5^2 + 10^2 = 25 + 100 = 125$$

So

$$c = \sqrt{125} = 11.2 \text{ inches}.$$
Areas of the pieces

Area of big square = $c^2$
Area of triangle = $\frac{1}{2}ab$
Area of little square = $(a - b)^2$

Areas of the assembled pieces

$4 \times \text{Area of Triangle} + \text{Area of little square} = \text{Area of big square}$

$4 \left(\frac{1}{2}ab\right) + (a - b)^2 = c^2$

$2ab + (a^2 - 2ab + b^2) = c^2$

$a^2 + b^2 = c^2$

**Example 5.** Inclines. A wheelchair ramp is to be installed over a set of 5 steps. The rise of each step is 6 inches, the length of each step is 10 inches. How long is the ramp?

Compare page 193, Problem 13.
The total rise of the steps is $5 \times 6 = 30$ inches.
The total length of the steps is $5 \times 10 = 50$ inches.

By the Pythagorean Theorem, the length of the ramp should be
$$\sqrt{30^2 + 50^2} = 58.3 \text{ inches}$$

**Example 6.** A baseball diamond is a square, 90 feet between bases. The shortstop is playing exactly halfway between second base and third. How far is the shortstop from first base?

Compare page 193, Problem 13.

**Solution:** Use the Pythagorean Theorem with side lengths $a = 90$ and $b = 45$:

$$c^2 = a^2 + b^2$$
$$c^2 = 90^2 + 45^2$$
$$c^2 = 8100 + 2025 = 10125$$
$$c = \sqrt{10125} = 100.6 \text{ feet}$$
Circles

The important theorems:

- Circumference = $2\pi \times$ radius
- Area = $\pi \times$ radius squared

Example 7. What is a better deal: a 10 inch pizza for $7 or a 14 inch pizza for $10?

Solution: The radius of the 10 inch pizza is $r = 5$. So the area of the small pizza is

$$A_{\text{small}} = \pi \cdot 5^2 = 25\pi$$

The radius of the 14 inch pizza is $r = 7$. So the area of the large pizza is

$$A_{\text{large}} = \pi \cdot 7^2 = 49\pi$$

Thus we see that the large pizza has almost twice the area, but only costs $10/7 = 1.4$ times as much. The large pizza is a much better deal.

Example 8. A windshield wiper rotates through a 120 angle as it cleans the windshield. The rubber part of the wiper blade is 14 inches long and connects to the wiper arm at a point 4 inches from the pivot point (and this extends to a point 18 inches from the pivot point). What is the area cleaned by the windshield wiper?

Solution: Since 120 is one third of 360, a complete circle, the tip of the 18 inch blade cleans an area equal to $\frac{1}{3}$ the area of a circle of radius $r = 14$:

$$A_{\text{tip}} = \frac{1}{3} \pi \cdot 18^2 = 339.3$$

From this area we need to subtract the area not swept out by the 4 inches at the bottom of the blade:

$$A_{\text{bottom}} = \frac{1}{3} \pi \cdot 4^2 = 16.8$$
So the final answer is

\[ A_{\text{cleaned}} = A_{\text{tip}} - A_{\text{bottom}} = 339.3 - 16.8 = 322.5 \text{ square inches} \]

**Volume Problem**

The volume of a box is \( \text{length} \times \text{width} \times \text{height} \).

**Example 9.** Your freezer compartment is 16 inches high, 20 inches deep, and 30 inches across. How many 4 by 3 by 2 inch cartons can you possibly fit in your freezer?

Compare page 195, Problem 26.

**Solution:** The volume of the freezer is

\[ V_{\text{freezer}} = 16 \times 20 \times 30 = 9600 \]

The volume of a single carton is

\[ V_{\text{carton}} = 4 \times 3 \times 2 = 24 \]

The total possible number you can fit in your freezer is:

\[ \frac{V_{\text{freezer}}}{V_{\text{carton}}} = \frac{9600}{24} = 400. \]

Note: You may not be able to fit in this exact number, because of wasted space while packing.