2. EXTRAPOLATING PERCENTAGES

- In 1989, Bryant Gumbel, then of the Today Show, interviewed Dr. Charles Hennekens, medical researcher at Harvard, who promoted recent finding confirming the long suspected benefits of aspirin in thwarting heart attacks.
  - The Doctor: Participants in the Harvard Study reduced their risk of heart attacks by 47% by taking one aspirin every other day.
  - Bryant Gumbel: Could you reduce the risk of a heart attack by 94% if you took an aspirin every day?

3. BRYANT GUMBEL CONT’D

- Let’s go one step further: If you took one and a half aspirin every day (3 times as much) would you reduce the risk by 141%?
- Note: A 47% reduction means that you now have .53 = 1 − .47 times the previous chance of a heart attack. So two such reductions would result in a risk factor of
  - $(.53)^2 = .28$
- Can you think of medical considerations?
4. Inflation

- Assuming 3 percent inflation over the next 20 years, how much will a 6000 motorcycle cost twenty years from now?
- Solution: \((1.03)^{20} \times 6000 = 10,837\)
- At 5 percent inflation, the value increases to \((1.05)^{20} \times 6000 = 15,920\)
- If you invest money in a 3 percent certificate of deposit, and the rate of inflation is greater than 3 percent, then you are losing money (in terms of buying power).

5. The Value of Education

The following table shows the average income of a person based on her or his level of education:

<table>
<thead>
<tr>
<th>Academic Level</th>
<th>1996 Income</th>
<th>Career Length</th>
<th>Career Earnings</th>
</tr>
</thead>
<tbody>
<tr>
<td>No diploma</td>
<td>16,536</td>
<td>46 years</td>
<td>1.70 million</td>
</tr>
<tr>
<td>H.S. diploma</td>
<td>23,400</td>
<td>44 years</td>
<td>2.37 million</td>
</tr>
<tr>
<td>Bachelor’s Deg</td>
<td>37,388</td>
<td>40 years</td>
<td>3.68 million</td>
</tr>
<tr>
<td>Advanced Deg</td>
<td>47,944</td>
<td>37 years</td>
<td>4.59 million</td>
</tr>
</tbody>
</table>

6. Value of Education

- Source: The Arithmetic of Life and Death by George Shaffner
- The data is adjusted to account for 5 percent inflation, increasing everyone’s salary accordingly.
• Assuming you spend six hours per day going to classes or studying, for 32 weeks a year for four years, how much is your Bachelor’s Degree worth to you, based on this chart?
• Solution: Do the math:

\[
\frac{3.68 - 2.37 \text{ million dollars}}{4 \text{ years} \times 32 \frac{\text{weeks}}{\text{year}} \times 5 \frac{\text{days}}{\text{week}} \times 6 \frac{\text{hours}}{\text{day}}} = 341 \text{ dollars/hour}
\]

7. GOING IN REVERSE

• Stereo speakers are marked “40% off”
• If the sale price is $100, what is the regular price of the speakers?
• Can you just add 60% of $100, to compensate for the 40% discount?
  – Since 60% of $100 is $60, this method says the original price was $160.
  – Let’s check. What is 40% off $160?
  – Answer: .6 \times 160 = 96, which is not $100.
  – So the original price was not $160.

8. GOING IN REVERSE

• Stereo speakers are marked “40% off” If the sale price is $100, what is the regular price of the speakers?
• Solution: If \( P \) is the original price, then we multiply \( P \) by .6 to get the sale price
• \( 0.6 \times P = 100 \)
• To find \( P \) we must divide:
• \( P = \frac{100}{0.6} = 166.67 \)
9. Sales Tax Example

- Assuming sales tax is 6.5%, what was the original price of a car if the price including the sales tax came to $13,312.50?
- **Solution:** If $P$ is the original price, then we multiply $P$ by 1.065 to get the final price including tax.
  - $1.065 \times P = 13312.50$
  - To find $P$ we must divide:
  - $P = \frac{13312.50}{1.065} = 12500$

10. Interest Example

- You invest a principal at 6% interest.
- If your account is worth $2650 after one year, how much did you initially invest?
- **Solution:** If $P$ is the original principal, then we multiply $P$ by 1.06 to get the amount in the account after one year.
  - $1.06 \times P = 2650$
  - To find $P$ we must divide:
  - $P = \frac{2650}{1.06} = 2500$

11. Compound Interest Example

- You invest a principal at 6% interest, compounded monthly.
- If your account is worth $3482.30 after one year, how much did you initially invest?
- **Solution:** If $P$ is the original principal, then we multiply $P$ by $(1 + \frac{0.06}{12})^{12} = 1.005^{12}$ to get the amount in the account after one year.
  - $1.005^{12} \times P = 3482.30$
  - To find $P$ we must divide:
\[ P = \frac{3482.30}{1.005^{12}} = 3280 \]

12. **Savings Bonds**

- How much does a $100 U.S. Savings Bond cost if it matures in five years at 4% interest, compounded annually?
- **Solution:** If \( P \) is the original price of the bond, then we multiply \( P \) by \( 1.04^5 \) to get the value of the bond after five years, which is guaranteed to be one hundred dollars:

\[ 1.04^5 \times P = 100 \]

- To find \( P \) we must divide:
- \( P = \frac{100}{1.04^5} = 82.19 \)
- The difference of $17.81 is the interest you earn for purchasing the bond.

13. **A Simple Error**

- A student from a previous semester once made a simple error in this calculation.
- Instead of writing
- \( P = \frac{100}{1.04^5} = 82.19 \)
- this student wrote
- \( P = \frac{100}{0.04^5} \)
- omitting the 1 in 1.04
- My calculator says
- \( \frac{100}{0.04^5} = 976, 562, 500 \)
- Does this answer seem reasonable?
14. Saving Money

- Suppose you put \( M \) dollars in the bank every month, for six months.
- If the bank pays 0.005 monthly interest (6% annual interest), your money grows like this:

<table>
<thead>
<tr>
<th>month</th>
<th>formula</th>
<th>calculated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.005(^5)M</td>
<td>1.0253M</td>
</tr>
<tr>
<td>2</td>
<td>1.005(^4)M</td>
<td>1.0202M</td>
</tr>
<tr>
<td>3</td>
<td>1.005(^3)M</td>
<td>1.0150M</td>
</tr>
<tr>
<td>4</td>
<td>1.005(^2)M</td>
<td>1.0100M</td>
</tr>
<tr>
<td>5</td>
<td>1.005(^1)M</td>
<td>1.0050M</td>
</tr>
<tr>
<td>6</td>
<td>( M )</td>
<td>( M )</td>
</tr>
<tr>
<td>Total</td>
<td>Sum</td>
<td>6.0755M</td>
</tr>
</tbody>
</table>

15. Final Balance

- After 6 months, you account will be worth

\[
1.005^5M + 1.005^4M + 1.005^3M + 1.005^2M + 1.005^1M + M
\]

- Factoring out an \( M \), this sum is

\[
(1.005^5 + 1.005^4 + 1.005^3 + 1.005^2 + 1.005^1 + 1)M
\]
- We want a formula for the sum inside the parentheses.

16. Math Saves the Day!

- Let \( x = 1.005 \) and
- We want to compute
- \( S = x^5 + x^4 + x^3 + x^2 + x + 1 \)
- Recall the formula
\[ 1 + x + x^2 + \cdots x^n = \frac{x^{n+1} - 1}{x - 1} \]

- Plugging in \( x = 1.005 \) and \( n = 5 \) gives
- \( S = \frac{x^6 - 1}{x - 1} \)
- Note that \( x - 1 = 1.005 - 1 = 0.005 \) is just the original monthly interest rate.

### 17. Final Calculation

- For our problem, if you deposit \( M = 100 \) dollars every month for six months, at a monthly rate of \( r = 0.005 \), you would have
- \( B = \frac{1.005^6 - 1}{0.005} \cdot 100 = 607.55 \) dollars after 6 months.
- The 7.55 represents accumulated monthly interest.

### 18. Monthly Savings Formula

\[ B = \frac{(1 + r)^n - 1}{r} M \]

where

- \( M = \) amount saved per month
- \( B = \) ending balance
- \( r = i/12 = \) monthly interest rate
- \( n = \) number of months

### 19. Long Term Savings

- Suppose you saved \( M = 100 \) dollars every month, for 45 years, at a monthly rate of \( r = 0.005 \).
8

- $M = 100$
- $r = .005$
- $n = 12 \times 45 = 540$

- After 45 years the account balance would be
- $B = \frac{1.005^{540} - 1}{.005}100 = 275,599$ dollars.
- Your monthly contributions were $540 \times 100 = 54,000$, so most of the growth in the account is due to accrued interest.

20. Saving for Retirement

- You and your sister have different ideas about saving for your retirement at age 65.
- At age 25 you start put aside 200 every month into an IRA yielding 7 percent interest.
- After ten years, at age 35, you decide that family obligations require you to save the $200 for college money for your three kids.
- So you stop putting money into the account, but let it continue to collect 7% interest until you retire at age 65.

21. Saving for Retirement Cont’d

- Your sister, on the other hand, waits until her 45th birthday to begin saving for her retirement.
- Like you, she has $200 taken out of her paycheck every month into a 7% IRA account.
- At age 65, who has saved more money: you or your sister?
22. Your Sister

- Let’s look at your sister first, since her situation is easier to analyze.
- The monthly interest rate is \( r = 0.07/12 \) for \( n = 240 \) months.
- Since \( M = 200 \), the monthly savings formula gives an ending balance after 20 years of
  \[
  \frac{(1 + 0.07/12)^{240} - 1}{0.07/12} 200 = 104,185
  \]
- more than doubling the \( 240 \times 200 = 48,000 \) she has invested.

23. Your Turn

- You on the other hand have invested one half as much money as your sister: \( 60 \times 200 = 24,000 \).
- After ten years, when you are 35, your ending balance is computed by the Monthly Savings Formula (with \( r = 0.07/12 \) and \( n = 120 \) months):
  \[
  B = \frac{(1 + 0.07/12)^{120} - 1}{0.07/12} 200 = 34,616.96
  \]
- Now this money earns compound interest over the next 30 years or \( 30 \times 12 = 360 \) months, so that at age 65 your IRA account will be worth
  \[
  (1 + 0.07/12)^{360} \times B = 280,968.48
  \]
- You made 2.7 times as much as your sister, although you only invested half as much money.

24. Advice

- Moral: How much you accumulate for retirement depends upon three things:
  \begin{itemize}
  \item (i) when you start saving,
  \item (ii) how much you manage to save, and
  \item (iii) how much your investments return over the long run.
  \end{itemize}
• Of the three, when you start saving turns out to be the most important.
• Moral: **Invest when you are young!**

### 25. Monthly Payment $M$ to Obtain Balance $B$

$$M = \frac{r}{(1+r)^n - 1}B$$

This formula comes from the Monthly Savings Formula

$$B = \frac{(1 + r)^n - 1}{r}M$$

### 26. Saving for College

• Kaylee’s parents want to put aside money every month so that their daughter will have $25,000 for college when she turns 18.
• At 5% annual interest, how much do they need to save per month?
• Set the variables:
  - amount saved per month: $M$
  - desired ending balance: $25,000$
  - monthly interest rate: $r = .05/12 = .004166667$
  - number of months: $12 \times 18 = 216$
• $M = \frac{.004166667}{(1.004166667)^{216} - 1}25,000 = 71.60$