1. Relative Weights

In terms of pounds,

<table>
<thead>
<tr>
<th>animal</th>
<th>average weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>elephant</td>
<td>9,000 pounds</td>
</tr>
<tr>
<td>bear</td>
<td>600 pounds</td>
</tr>
<tr>
<td>human</td>
<td>180 pounds</td>
</tr>
</tbody>
</table>

Thus an elephant weighs 15 times more than a bear
a bear weighs $3\frac{1}{3}$ times more than a human

Question: How much more does an elephant weigh than a human?

Answer: Multiply: An elephant weighs

$$15 \times \frac{10}{3} = 50$$
times as much as a human

2. Relative Speeds

In terms of mph (miles per hour),

<table>
<thead>
<tr>
<th>vehicle</th>
<th>average speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>airplane</td>
<td>600 mph</td>
</tr>
<tr>
<td>car</td>
<td>60 mph</td>
</tr>
<tr>
<td>bicycle</td>
<td>15 mph</td>
</tr>
<tr>
<td>walker</td>
<td>3 mph</td>
</tr>
</tbody>
</table>

Thus a plane travels 10 times faster than a car
a car travels 4 times faster than a bike
a bike travels 5 times faster than a person walking

Question: How much faster is traveling in an airplane compared to walking?
Answer: Multiply: An airplane travels

\[ 10 \times 4 \times 5 = 200 \]
times as fast as a person on foot.

3. **The Moral**

The moral to these two examples is that

**Rates Multiply!**

This is a famous rule of calculus, called the chain rule which says
If we have three variable \( x, y, \) and \( z, \)
If \( z \) is changing \( m \) times faster than \( y \)
and \( y \) is changing \( n \) times faster than \( x, \)
then \( z \) is changing \( m \cdot n \) times faster than \( x. \)

4. **The Chain Rule**

Since the derivate tells us the rate of change, the fact that rates multiply can be written succintly as

\[
\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}
\]

This formula is called the **Chain Rule**

When measuring weights, our first example becomes

\[
\frac{d\text{elephant}}{d\text{human}} = \frac{d\text{elephant}}{d\text{bear}} \cdot \frac{d\text{bear}}{d\text{human}}
\]
5. The Chain Rule Continued

When measuring speed, the second example shows that you can string several rates together:

\[
\frac{d\text{ plane}}{d\text{ pedestrian}} = \frac{d\text{ plane}}{d\text{ car}} \cdot \frac{d\text{ car}}{d\text{ bike}} \cdot \frac{d\text{ bike}}{d\text{ pedestrian}}
\]

which illustrates why the rule is called the chain rule.

6. Using the Chain Rule

Suppose

\[ z = y^2 \text{ and } y = x^3 + 11 \]

Use the chain rule to find \( \frac{dz}{dx} \)

\[
\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}
\]

\[
= 2y \cdot 3x^2 = 2(x^3 + 11) \cdot 3x^2
\]

7. Direct Calculation

\[ z = y^2 \text{ and } y = x^3 + 11 \]

So \( z = (x^3 + 11)^2 = x^6 + 22x^3 + 121 \)

So \( \frac{dz}{dx} = 6x^5 + 66x^2 \)

This is equivalent to our previous solution:

\[
\frac{dz}{dx} = 2(x^3 + 11) \cdot 3x^2 = 6x^3(x^3 + 11)
\]

\[
= 6x^6 + 66x^3
\]
8. Prime Notation Version

Suppose we combine two functions \( f \) and \( g \) to get

\[ f \circ g(x) = f(g(x)) \]

Then the Chain Rules becomes

\[ \frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x) \]

In words this says: the derivative of a composition is the derivative of the outer function evaluated at the inner functions times the derivative of the inside part.

9. Examples

Suppose \( f(x) = x^2 \) and \( g(x) = x^3 + 11 \)

Then \( f'(x) = 2x \) and \( g'(x) = 3x^2 \)

So \( \frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x) \)

\[ = 2g(x) \cdot 3x^2 \]

\[ = 2(x^3 + 11) \cdot 3x^2 \]

We have seen this example before.

10. Second Example

Differentiate \( y = \sqrt{7x^3 + x^2 + 1} \) and We are using two functions:
\( f(x) = \sqrt{x} \) and \( g(x) = 7x^3 + x^2 + 1 \).

Since \( f'(x) = \frac{1}{2}x^{-1/2} \) and \( g'(x) = 21x^2 + 2x \)

So \( \frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x) \)

\[ = \frac{1}{2}(7x^3 + x^2 + 1)^{-1/2} \cdot (21x^2 + 2x) \]

11. Chaining the Chain Rule

Differentiate \( f(x) = \sqrt{x + \sqrt{x + 1}} \)
Here we must use the chain rule twice.

The secret is to write the square roots as exponents:

Differentiate $f(x) = (x + (x + 1)^{\frac{1}{2}})^{\frac{1}{2}}$

\[
f'(x) = \frac{1}{2} \left( x + (x + 1)^{\frac{1}{2}} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left( x + (x + 1)^{\frac{1}{2}} \right)
\]

\[
f'(x) = \frac{1}{2} \left( x + (x + 1)^{\frac{1}{2}} \right)^{-\frac{1}{2}} \cdot \left( 1 + \frac{d}{dx} (x + 1)^{\frac{1}{2}} \right)
\]

\[
f'(x) = \frac{1}{2} \left( x + (x + 1)^{\frac{1}{2}} \right)^{-\frac{1}{2}} \cdot \left( 1 + \frac{1}{2} (x + 1)^{-\frac{1}{2}} \right)
\]

12. The Rules Combined

Differentiate $f(x) = x\sqrt{2x + 1}$

By the product rule, chain rule, and power rule,

\[
f'(x) = 1 \cdot \sqrt{2x + 1} + x \cdot \frac{d}{dx} (2x + 1)^{1/2}
\]

\[
= \sqrt{2x + 1} + x \cdot \frac{1}{2} (2x + 1)^{-1/2} \cdot \frac{d}{dx} (2x + 1)
\]

\[
= \sqrt{2x + 1} + \frac{1}{2} x (2x + 1)^{-\frac{1}{2}} \cdot 2
\]

\[
= \sqrt{2x + 1} + x (2x + 1)^{-\frac{1}{2}}
\]

13. The Rules Combined II

Differentiate $f(x) = \sqrt[3]{\frac{x}{x + 1}}$

By the chain rule, power rule, and quotient rule,

\[
f'(x) = \frac{1}{2} \left( \frac{x}{x + 1} \right)^{-1/2} \cdot \frac{d}{dx} \left( \frac{x}{x + 1} \right)
\]
\[
\frac{1}{2} \left( \frac{x}{x+1} \right)^{-\frac{1}{2}} \cdot \frac{d}{dx} \left( x(x+1)^2 \right) - x \cdot \frac{d}{dx} (x+1) \cdot (x+1)^{-1} \\
= \frac{1}{2} \left( \frac{x}{x+1} \right)^{-\frac{1}{2}} \cdot \frac{1(x+1) - x(1)}{(x+1)^2} \\
= \frac{1}{2} \left( \frac{x}{x+1} \right)^{-\frac{1}{2}} \cdot \frac{1}{(x+1)^2}
\]