A Foundation for Geometry

There is no royal road to geometry – Euclid

1. Points and Lines

We are ready (finally!) to talk about geometry. Our first task is to say something about points and lines. We will assume that we have a (nonempty) space $X$ and that the points in our geometry are just elements of $X$. There are no restrictions (for now) on the set $X$. It could be a finite set, the real number line, the $x-y$ plane, or even three dimensional space. We assume further that we have a set $\mathcal{L}$ of subsets of $X$. The elements of $\mathcal{L}$ are called the lines of our geometry.

As with metric spaces we need some axioms or rules about the possible points and lines of our geometry. The first three rules are called the incidence axioms. They are very basic.

**Incidence Axioms**

Axiom 1. There are at least two different lines in $\mathcal{L}$.

Axiom 2. Each line contains at least two different points.

Axiom 3. For any two distinct points $A$ and $B$, there is exactly one line $\ell$ in $\mathcal{L}$ which contains them.

The third axiom is often stated as “two points determine a line” and dates back to Euclid. The unique line determined by $A$ and $B$ is written $\overrightarrow{AB}$. Points which lie on the same line are called **collinear**.

Our first theorem in geometry is an easy consequence of Axiom 3

**Theorem 4.1.** Distinct lines $m$ and $n$ intersect in at most one point
4.1 Exercises

1. Prove Theorem 4.1.

2. Prove that there must be at least three points in $X$.

2. Distance and Betweenness

We will assume that our space has a distance function $d$ defined on it which obeys the four distance axioms of a metric space.

It is convenient to use a simplified notation for the distance between two points. Assuming that we are using the distance function $d$, then we shall denote the distance $d(A, B)$ between two points $A$ and $B$ by

$$AB \overset{\text{def}}{=} d(A, B).$$

We use the topological axioms of distance as a springboard into geometry. In terms of this simplified notation, the four distance axioms can be stated.

Distance Axioms

Axiom 1. $AB \geq 0$ [non-negativity]

Axiom 2. $AB = 0$ if and only if $A = B$ [zero property]

Axiom 3. $AB = BA$ [symmetry]

Axiom 4. $AC \leq AB + BC$ [triangle inequality]

Remark. Most authors do not include Axiom 4, the triangle inequality, since it is possible to prove the triangle inequality from the other axioms of geometry. Since we are not critically interested in developing a minimal set of axioms here, it suits our purpose to assume a priori that our distance function $d$ satisfies the triangle inequality.

Given three distinct points $A$, $B$, or $C$ on a line, we often need to know in which order they appear. In general we expect that one of them will lie between the other two. But the question is: which one is in the middle? The Greek geometers did not define the concept of “betweenness,” relying on diagrams instead.
We will use the notation $A - B - C$ to designate that $A$, $B$, and $C$ are collinear points and that $B$ is between $A$ and $C$.

We can use distance to define **betweenness**, as follows:

**Definition 4.1.** $A - B - C$ if and only if $A$, $B$, $C$ are three distinct collinear points and

\[ AB + BC = AC. \]

The following theorem formalizes the intuitively obvious statement that if three points line up in order $A, B, C$, then going backwards they are in order $C, B, A$.

**Theorem 4.2.** $A - B - C$ implies $C - B - A$.

The definition of betweenness does not assert that a betweenness relation must hold between three collinear points. It merely defines the condition that must be met if a betweenness relation actually occurs. The existence of a betweenness relation requires another axiom.

**Betweenness Axiom.** Given three distinct collinear points. At least one betweenness relation $A - B - C$, $A - C - B$, or $B - A - C$ holds.

The next theorem says that only one of the points $A$, $B$, or $C$ can lie in the middle. For this reason, we call this the “unique middle” theorem.

**Theorem 4.3.** [unique middle] If $A - B - C$ is true, then $A - C - B$ and $B - A - C$ are both false.

### 4.2 Exercises

1. Prove Theorem 4.2.
2. Prove Theorem 4.3.

### 3. Examples of Geometries

**Example 4.1.** [The Euclidean plane] Let the space $X$ be the $x$–$y$ plane \{(x, y) : x and y are real numbers\}. The lines in $\mathcal{L}$ are sets of the form

\[ \{(x, y) : ax + by = c\}, \]
where \( a, b, \) and \( c \) are real numbers, not all zero. The usual definition of distance between two points \( P = (x_1, y_1) \) and \( Q = (x_2, y_2) \) in the plane is
\[
d(P, Q) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.
\]

**Example 4.2** (The \( x-y \) plane with the taxicab metric). \( X \) and \( L \) are defined as in Example 4.1. The distance between two points \( P = (x_1, y_1) \) and \( Q = (x_2, y_2) \) in the plane is
\[
d(P, Q) = |x_1 - x_2| + |y_1 - y_2|.
\]

**Example 4.3** (Lines on a sphere). Let \( X \) be a sphere and suppose that we restrict travel to the surface of the sphere. Then the distance between two points \( P \) and \( Q \) on the sphere is measured along the great circle of the sphere which passes through the two points. A great circle is a circle on the sphere whose center is the center of the sphere. Note that because of a technicality, the third incidence axiom does not hold on a sphere. Specifically, if \( A \) is the North Pole and \( B \) is the South Pole, then any longitudinal line passes through these two points. It is possible to modify the incidence axioms to allow for this exception.

**Example 4.4** (Discrete space). Let \( X \) be any set with at least three elements. The set \( L \) consists of all possible two element sets \( \{a, b\} \) where \( a \) and \( b \) lie in \( X \). The following function is a distance function:
\[
d(a, b) = \begin{cases} 
0 & \text{if } a = b \\
1 & \text{if } a \neq b.
\end{cases}
\]
Note that it is impossible to find three distinct points on any line in this model, so no betweenness relations hold.

**Example 4.5** (The Fano Plane). The Fano plane \( \mathcal{F} \) is a collection of 7 points
\[
X = \{A, B, C, D, E, F, G\}
\]
and 7 lines, determined by the drawing in Figure 1.

The seven lines are

(1) \( A, B, D \)
(2) \( A, C, E \)
(3) \( B, C, F \)
(4) \( A, F, G \)
(5) \( B, E, G \)
(6) \( C, D, G \)
(7) \( D, E, F \)
3. EXAMPLES OF GEOMETRIES

The solid and dotted lines in the figure are only to aid visualization of the lines. There are only seven points and seven lines in the Fano plane. The Fano plane is also called the **projective plane** of order 2.

Distances are assigned so that the distance function satisfies the zero property and symmetry, that is, the distance from a point to itself is 0 and the distance from point \( x \) to point \( y \) is the same as the distance from point \( y \) to point \( x \). Six distances are defined to be 2:

\[
AB = AC = AF = BE = BC = CD = 2
\]

The remaining distances \( d(x, y) \), when \( x \neq y \), have the value 1.

4.3 Exercises

1. On a sphere of radius 13, what is the distance between \( A = (0, 0, 13) \) and \( (3, 4, 12) \)?

2. We have defined lines in the \( x-y \) plane with the standard distance function to be the same as lines in the \( x-y \) plane using the taxicab metric. Show that if \( A, B, \) and \( C \) are three distinct points on a line in the \( x-y \) plane, then the betweenness relation \( A-B-C \) holds using the standard distance function if and only if \( A-B-C \) holds using the taxicab metric.

3. What are the betweenness relations for the discrete space?
4. Ignoring symmetry, there are six betweenness relations in the Fano Plane. Find them.

5. Show that the Fano plane $\mathcal{F}$ has the following interesting property:

**Property (*)&:** Any line passing through any two points in $\mathcal{F}$ passes through a third point in $\mathcal{F}$.

You will never find a finite collection of *non-collinear* points in the Euclidean $x$–$y$ plane that satisfy Property (*).

6. The Fano plane has a remarkable coloring property. To warm up on an easier example, consider the following three pairs of letters: (1) \{A, B\}, (2) \{A, C\}, and (3) \{B, C\}. Now suppose you color each letter, using exactly two colors, red and blue. Each letter must be colored the same in every pair it appears. For example, if you color $A$ in the first pair red, then you must color $A$ in the second pair red also. Explain why, no matter how the three letters are colored, at least one of the three pairs will have the same color.

Suppose we switch from pairs to triples. Can we construct an example similar to the three pairs displayed above? The key is to use the Fano plane.

Show that no matter how you color the seven points in the Fano plane $\mathcal{F}$ using two colors (red and blue), it will always be the case that at least one of the seven lines will contain three points all of the same color.

### 4. Segments and Rays

We can use the notion of betweenness to define segments and rays in our geometry. Suppose $X$ is a set of points, $d$ is a distance function on $X$, and $\mathcal{L}$ is the set of lines.

**Definition 4.2** (line segment $\overline{AB}$). If $A$ and $B$ are distinct points in $X$, then

$$\overline{AB} \overset{\text{def}}{=} \{A, B\} \cup \{P : A - P - B\}.$$ 

The points $A$ and $B$ are called the endpoints of the segment $\overline{AB}$. A segment consists of its endpoints and the points in between them. The points strictly between $A$ and $B$ are called the interior points of $\overline{AB}$.

**Definition 4.3** (ray $\overrightarrow{AB}$). If $A$ and $B$ are distinct points in $X$, then

$$\overrightarrow{AB} \overset{\text{def}}{=} \overline{AB} \cup \{P : A - B - P\}.$$
The starting point $A$ is called the endpoint of ray $\overrightarrow{AB}$. The points on $\overrightarrow{AB}$ other than $A$ are called the interior points of $\overrightarrow{AB}$.

The following three theorems, which are intuitively obvious if you draw diagrams, can be proven readily using our definitions of line segment and ray.

**Theorem 4.4.** $\overrightarrow{AB} = \overrightarrow{BA}$

**Theorem 4.5.** $\overrightarrow{AB} \cap \overrightarrow{BA} = \overrightarrow{AB}$

**Theorem 4.6.** $\overrightarrow{AB} \cup \overrightarrow{BA} = \overrightarrow{AB}$

We would like to have an axiom which says that our rays “look like” rays on the real numbers line. This is accomplished by the following axiom.

**The Ruler Axiom.** Given a ray $\overrightarrow{AB}$ and a real number $t \geq 0$. Then there is a point $X$ on $\overrightarrow{AB}$ such that $AX = t$.

The Ruler Axiom can be used to prove the following theorem about midpoints:

**Theorem 4.7.** [midpoint] Given segment $\overrightarrow{AB}$ there is a unique point $M$ on $\overrightarrow{AB}$, called the midpoint of $\overrightarrow{AB}$, such that $AM = MB$.

### 4.4 Exercises

1. Write an equation of the form $y = mx + b$ to describe the segment $\overrightarrow{AB}$ in $\mathbb{R}^2$ with $A = (0, 0)$ and $B = (1, 2)$.

2. Let $\ell$ be the line $y = 2x + 1$ in the $x$-$y$ plane. (a) Find two point on $\ell$ whose Euclidean distance from the point $(0, 1)$ is precisely 5. (b) Find two point on $\ell$ whose taxicab distance from the point $(0, 1)$ is precisely 5.

3. Is it possible to find two points $X_1$ and $X_2$ on $\overrightarrow{AB}$ whose distance from $A$ is $t$?

4. Is it possible to find two points $X_1$ and $X_2$ on $\overrightarrow{AB}$ whose distance from $A$ is $t$?


5. Dimension

If we have a betweenness relation \( B \prec A \prec C \) then the rays \( \overrightarrow{AB} \) and \( \overrightarrow{AC} \) are called \textbf{opposite} rays. Notice that the ray \( \overrightarrow{AB} \) together with the opposite ray \( \overrightarrow{AC} \) comprise the entire line \( \overline{AB} \). Since the two rays meet only at the endpoint \( A \), we see that a point \( A \) on a line \( \ell \) separates the line into three disjoint pieces:

1. the point \( A \) itself
2. the interior of the ray \( \overrightarrow{AB} \) lying on \( \ell \)
3. the interior of the opposite ray \( \overrightarrow{AC} \) lying on \( \ell \).

In a similar way a line \( \ell \) separates the plane into three disjoint pieces:

1. the line \( \ell \) itself
2. a halfplane on one side of \( \ell \)
3. the opposite halfplane on the other side of \( \ell \).

This separation fact is one of the basic axioms used in studying the foundations of geometry. Rather than try to state a precise definition of the separation axiom, we will just contend ourselves with this informal description.

It is separation by lines that makes our geometry two dimensional. Consider the three dimensional world we live in. To locate a point in our world we must specify three values: width, depth, and height. All the other axioms of geometry hold in three dimensional space: two points determine a line; the Betweenness Axiom; the Ruler Axiom. Segments and rays can be defined in three space using the notion of betweenness. A line, however, does not separate three dimensional space into three disjoint pieces. Separation in three space requires a plane, which acts like a wall, dividing the space into the wall itself and the two regions on either side of the wall.

The idea of dimension works as follows. A point has dimension 0; it has no width, depth, or height. A line is one dimensional; it can be separated into three pieces by a 0 dimensional point. A plane has two dimensions; it can be separated by a one dimensional line. Three dimensional space is separated by a two dimensional plane.
4.5 Exercises

1. Consider the \(x - y\) plane and take \(\ell\) to be the line \(y = 2x + 1\). Describe the two halfplanes separated by \(\ell\).

2. Consider the sphere of the earth. What “halfplanes” are separated by the equator?

3. Given two random lines in three dimensional space, do you think it is very likely that they intersect?

4. Think of the fourth dimension as time. Explain how a three dimension subspace of space-time at time \(t = 0\) separates space-time into three distinct parts.

6. Angles

Two rays \(a\) and \(b\) are called coterminal if they have the same endpoint. If this common endpoint is \(A\), then there must be points \(B\) and \(C\) such that

\[
a = \overrightarrow{AB} \quad \text{and} \quad b = \overrightarrow{AC}.
\]

The union of the two coterminal rays \(\overrightarrow{AB}\) and \(\overrightarrow{AC}\) is called the angle \(\angle BAC\).

\[
\begin{array}{c}
\text{A} \\
\text{b} \\
\text{C}
\end{array}
\]

Each angle has a measure \(m\angle BAC\). We also write \(m\angle BAC\) as \(m(a, b)\) to indicate that angle measure is a kind of “distance” between two coterminal rays \(a\) and \(b\). Angle measurement must satisfy the following four properties:

<table>
<thead>
<tr>
<th>Angle Measurement Axioms</th>
</tr>
</thead>
<tbody>
<tr>
<td>Axiom 1. (0 \leq m(a, b) \leq 180).</td>
</tr>
<tr>
<td>Axiom 2. (m(a, b) = 0) if and only if (a = b). [zero property]</td>
</tr>
</tbody>
</table>
Axiom 3. \( m(a, b) = 180 \) if and only if \( a \) and \( b \) are opposite rays.

Axiom 4. \( m(a, b) = m(b, a) \) [symmetry].

**Example 4.6.** Consider the model of the \( x-y \) plane. Given any pair of rays \( a \) and \( b \) whose common endpoint is the origin (0,0). Each pair of rays \( a \) and \( b \) will intersect the unit circle

\[
x^2 + y^2 = 1
\]

at unique points \( A \) and \( B \). We define the angle measure \( m(a, b) \) to be the length of the shortest arc connecting \( A \) and \( B \) times \( \frac{180}{\pi} \).

![Angle measurement diagram]

Given three coterminal rays \( a, b, \) and \( c \), we use angle measurement to define precisely what we mean when we say that ray \( b \) is between rays \( a \) and \( c \), written \( a - b - c \).

**Definition 4.4.** \( a - b - c \) if and only if \( a, b, c \) are three distinct coterminal rays and

\[
m(a, b) + m(b, c) = m(a, c).
\]

**Betweenness Axiom for Rays.** If three distinct coterminal rays \( a, b, \) and \( c \), all lie in the same closed halfplane whose edge contains one of them, then a betweenness relation \( a - b - c, a - c - b, \) or \( b - a - c \) holds.

**Theorem 4.8.** [unique middle for rays] If \( a - b - c \) is true, then \( a - c - b \) and \( b - a - c \) are both false.

Notice the similarity between the definition of betweenness of points and betweenness of rays. We can easily modify our proof of the Unique Middle Theorem (Theorem 4.3) for collinear points to prove the Unique Middle Theorem for coterminal rays. All we do is “search and replace” the words
6. ANGLES

- point $\rightarrow$ ray
- collinear $\rightarrow$ coterminal
- upper case $A, B, C \rightarrow$ lower case $a, b, c$

Finally, we need an axiom which says we can make angles of any measure between 0 and 180. One technicality is that we want to be able to construct our angle on either side of the line containing one of the sides of the angle.

**Protractor Axiom.** Given a ray $a$, a halfplane $H$ whose edge contains $a$, and a real number $s$ in the interval $0 \leq s \leq 180$. Then there is a ray $r$ with the same endpoint as $a$ lying in the halfplane $H$ such that

$$m(a, r) = s.$$

**Some terminology:** Angle $\angle ABC$ is a(n)

- **right** angle if $m\angle ABC = 90$
- **acute** angle if $0 < m\angle ABC < 90$
- **obtuse** angle if $90 < m\angle ABC < 180$
- **zero** angle if $m\angle ABC = 0$
- **straight** angle if $m\angle ABC = 180$
- **proper** angle if $0 < m\angle ABC < 180$

One way to remember how to distinguish an acute angle from an obtuse angle is to recall that in English, “acute” means sharp (an acute pain is a sharp pain) while “obtuse” means dull (an obtuse fellow is a dull person). An arrowhead whose tip forms an acute angle is sharp and dangerous, while an arrowhead whose tip forms an obtuse angle is not likely to penetrate skin.

The Protractor Axiom can be used to prove the following theorem about angle bisectors:

**Theorem 4.9.** [angle bisector] Given a nonzero angle $\angle BAC$. There is a unique ray $\overrightarrow{AD}$ between $\overrightarrow{AB}$ and $\overrightarrow{AC}$, called the **angle bisector** of $\angle BAC$, such that

$$m \angle BAD = m \angle DAC.$$
4.6 Exercises

1. Given three coterminal rays \( a, b, \) and \( c, \) is it always the case that at least one ray is between the other two? Hint: choose the rays so that \( m(a, b) = m(a, c) = m(b, c) = 120. \)

2. Is the ray \( x \) in the protractor axiom unique?


5. How can we define the interior of an angle \( \angle BAC? \) Is it true that an angle separates the plane into three distinct parts: (i) the angle itself, (ii) the interior of the angle, and (iii) the exterior of the angle?

7. Triangles

Professor Flappenjaw: Today we’re studying triangles. Who can tell me what a triangle is?

Ernest: You need three points \( A, B, \) and \( C. \)

Professor: Is a triangle just these three points?

Hedda: No, you gotta connect them to each other.

*She draws on the board:*

```
A ---- B ---- C
```

Professor: I see. So the triangle \( \triangle ABC \) is just the union of the three segments \( AB \cup BC \cup AC. \)

Ernest: Sweet.

Professor: And you could use any three points to make a triangle?
Dana: As long as they’re distinct, why not?

Professor: What if the three points all lie on the same line?

Ernest: You’d get a really flat triangle.

Professor: Do you want to even call such a thing a triangle?

Dana: [grudgingly] I guess not.

Professor: Triangle $\triangle ABC$ is $\overline{AB} \cup \overline{BC} \cup \overline{AC}$, provided that $A$, $B$, and $C$ are distinct and non-collinear. Each of the three noncollinear points is called a vertex of the triangle. Each segment is called a side of the triangle and each of the angles $\angle BAC$, $\angle ABC$, and $\angle ACB$ is called an angle of the triangle. These are often written more simply as $\angle A$, $\angle B$, and $\angle C$.

Ernest: It’s easy to identify the angles. Angle $A$ is the angle at vertex $A$. But it’s harder to identify the sides.

Professor: Maybe this will help. Take triangle $ABC$. We call side $\overline{BC}$ the side opposite $\angle A$. Notice that the letters $B$ and $C$ are the two vertices of the triangle other than $A$.

Hedda: So to find the side opposite angle $C$, we delete $C$ from the vertices $A, B, C$ to get segment $\overline{AB}$.

Professor: Exactly. Conversely, angle $\angle A$ is the angle opposite side $\overline{BC}$.

Hedda: In this case $A$ is the letter missing from the endpoints of the segment, $B$ and $C$.

Professor: Right. Tell me, can the side lengths be any three numbers we choose?

Ernest: As long as they’re positive, why not?

Professor: Can you draw a triangle with sides of length 2, 3, and 7?

Matthew: No way, the 7 is too large.

Carole: I see what the problem is. The triangle inequality says that the sum of two sides—in this case $2 + 3 = 5$—must always be larger than the third side.

Ernest: That’s why it’s called the triangle inequality.
Professor: Exactly. If I have three positive real numbers whose values do not violate the triangle inequality, that is, the sum of any two of them is greater than the third, then can I always draw a triangle whose lengths are these three numbers?

Ernest: I can’t see any reason why you can’t, but I can’t see any way to prove why you always can.

Professor: Let me ask a slightly different question. Suppose I wanted a triangle \( \triangle ABC \) where the measure of angle \( \angle B \) is, say 67, and the lengths \( AB \) and \( BC \) are 7 and 29. Can you always draw such a triangle?

Hedda: Well, you could use the Protractor Axiom to construct the 67 degree angle.

Dana: And then you could use the Ruler Axiom to make the sides the required lengths.

Ernest: Thank God for Axioms.

Professor: Here’s my final question to think about for the next class. Suppose that Dana and Carole each make a triangle whose sides have length 7 and 29, where the in-between angle measures 67. Will their two triangles look exactly the same?

4.7 Exercises

1. Draw a triangle \( \triangle ABC \). Now draw a triangle \( \triangle CDE \), where \( E \) lies on \( AC \) and \( C – B – D \). Which angle of \( \triangle CDE \) is congruent to an angle of \( \triangle ABC \).

2. Try do this problem without drawing a picture. Fill in the blanks. In triangle \( \triangle PQR \),
   (a) segment \( RQ \) is opposite ________.
   (b) angle \( \angle PRQ \) is opposite ________.
   (c) segment \( QP \) is opposite ________.

8. Congruence

Professor: Okay, class, today we are discussing congruence. What does it mean to say that two geometric figures are congruent?

Matthew: It means they have the same size and shape.

Professor: What does this mean, precisely.
Dana: That you could trace the first one and then lay the tracing over the second figure and the two would perfectly coincide.

Professor: Very good. In mathematical terms, laying a tracing of one figure over another can be thought of as subjecting it to a rigid motion. You can translate it, rotate it, reflect it, but you cannot bend, stretch, or shrink it.

Carole: Kind of the opposite of the rules in topology.

Professor: Good observation, Carole. I do have a specific figure in mind. What does it mean to say that two segments $\overline{AB}$ and $\overline{XY}$ are congruent?

Ernest: They have the same length?

Professor: Right. And what does it mean to say that two angles $\angle A$ and $\angle X$ are congruent?

Hedda: They have the same angle measure.

Professor: Correct. Now what does it mean to say that two triangles $\triangle ABC$ and $\triangle XYZ$ are congruent?

Dana: That’s a lot harder. It means that corresponding sides are congruent and corresponding angles are congruent.

Ernest: C. P. C. T.

Professor: What does that stand for?

Ernest: We learned it in high school. It means congruent parts of corresponding triangles.

Hedda: Don’t you mean corresponding parts of congruent triangles?

Ernest: Whatever.

Professor: Getting back to triangles $\triangle ABC$ and $\triangle XYZ$, how do you know what side of $\triangle XYZ$ corresponds to, say, side $\overline{AC}$ of $\triangle ABC$?

Dana: You pair the vertices in the order you write them. Since you wrote the vertices of the first triangle in the order $A, B, C$ and the vertices of the second triangle in the order $X, Y, Z$, the pairing is $A$ with $X$, $B$ with $Y$, and $C$ with $Z$. So side $\overline{AC}$ of the first triangle corresponds to side $\overline{XZ}$ of the second.
Professor: Are you suggesting that in order to show that \( \triangle ABC \) is congruent to \( \triangle XYZ \) we must show
\[
AB = XY \quad \quad m\angle A = m\angle X \\
BC = YZ \quad \quad \text{and} \quad m\angle B = m\angle Y \\
CA = ZX \quad \quad \quad \quad \quad \quad m\angle C = m\angle Z
\]
Ernest: That’s a lotta work, Professor.

Matthew: There’s gotta be an easier way.

Professor: And there is. Can any one tell me what it is?

Matthew: Something about Side – Angle – Side.

Ernest: I’ve heard of that, but I never really understood it.

Dana: It means that if you match up two sides
\[
AB = XY \quad \quad \quad \quad \quad AC = XZ
\]
and the angle in between these sides
\[
m\angle A = m\angle X,
\]
then you know the two triangles are congruent, without having to check the other side or the other two angles.

Hedda: That saves half the work.

Professor: How do you know this always works?

Ernest: It says so it our book.

Professor: Why does the book say so?

Hedda: Because it’s a true mathematical fact.

Professor: As a fact, is it a theorem or is it an axiom?

Ernest: Huh?

Professor: I mean, can you prove the Side–Angle–Side Statement using the other axioms of geometry or must we accept it as an axiom?

Matthew: It seems a lot more complicated that our other axioms.
Professor: Yes, it does. As it turns out, however, it is an axiom of geometry. You can’t really prove it from the other axioms.

Carole: How can you be sure? Maybe there’s a proof that is really hard and you just haven’t thought of it.

Professor: Let me explain. Suppose you could prove Side-Angle-Side from the other axioms. The x-y plane with the taxi-cab distance turns out to satisfy all these axioms, so it would have to obey Side-Angle-Side. But it doesn’t.

Ernest: No lie?

Professor: Really. You can convince yourself. Here’s a hint. Study the following two triangles:

```
A
  |
  |
  |
  |
  |
C
B
```

```
X
Y
Z
```

The coordinate are:

\[ A = (0,0) \quad B = (0,1) \quad C = (1,0) \]

and

\[ X = (0,0) \quad B = \left( \frac{1}{2}, \frac{1}{2} \right) \quad C = \left( \frac{1}{2}, \frac{1}{2} \right) \]

**Do-It-Now Exercise.** Using the taxicab distance function, show that

(a) \( AB = XY \)

(b) \( AC = XZ \)

(c) \( m\angle A = m\angle X \)

Does \( BC = YZ \)?  □
The Professor’s example demonstrates that Side–Angle–Side is actually an axiom—not a theorem—of geometry.

**Side–Angle–Side (SAS) Axiom.** Given a correspondence between triangles $\triangle ABC$ and $\triangle XYZ$. If (i) $AB = XY$, (ii) $AC = XZ$, and (iii) $m\angle A = m\angle X$, then $\triangle ABC$ is congruent to $\triangle XYZ$.

In words, if two sides of one triangle are congruent to the corresponding sides of a second triangle, and if the angles between these two sides have the same measure, then the two triangles are congruent.

Finally, a word about notation. The traditional symbol for “is congruent to” is “$\cong$.” The statement “angle $A$ is congruent to angle $B$” can be written as “$\angle A \cong \angle B$”, or equivalently, as the equation $m\angle A = m\angle B$.” The statement $\angle A = \angle B$, on the other hand, asserts that the two angles are identical, not just in measure, but in their location.

### 4.8 Exercises

1. Does Side–Angle–Side hold on the surface of a football? Measure the hypotenuse of a right triangle drawn in the middle, with one leg along the seam, and the hypotenuse of a right triangle drawn at the end of the football.

2. Given: $\triangle ABC$

   $AB = AC$

   the angle bisector of $\angle A$ crosses $BC$ at point $M$

   Prove: $M$ is the midpoint of $BC$.

3. [Construct the Segment] Given a proper angle $\angle BAC$ and a point $P$ in the interior of this angle. (This means that $P$ and $C$ are on the same side of $\overrightarrow{AB}$ and that $P$ and $B$ are on the same side of $\overrightarrow{AC}$.) Show how to use the our axioms to obtain points $X$ and $Y$ such that

   (i) $X$ is on $\overrightarrow{AB}$

   (ii) $Y$ is on $\overrightarrow{AC}$

   (iii) $P$ is the midpoint of $XY$

   [Hint: Use the Ruler Axiom to locate a point $Q$ on $\overrightarrow{AP}$ such that $AQ = 2AP$.]
9. Isosceles Triangles and Triangle Congruence Theorems

An easy and instructive application of Side–Angle–Side is in the proof of the Isosceles Triangle Theorem.

**Definition 4.5.** An **isosceles** triangle is a triangle which has at least two congruent sides. An **equilateral** triangle has all three sides congruent. A triangle in which all three lengths are different numbers is called **scalene**.

**Definition 4.6.** A **right** triangle is a triangle, one of whose angles is a right angle. The side opposite the right angle is called the **hypotenuse** of the right triangle; the other two sides are called the **legs** of the triangle.

A well known fact from high school geometry is

**Theorem 4.10.** [Isosceles triangle theorem] Given \( \triangle ABC \). If \( AB = AC \), then \( m\angle B = m\angle C \).

This theorem asserts that the corresponding angles of an isosceles triangle are congruent.

To prove this theorem, we set up the following 1–1 correspondence between \( \triangle ABC \) and \( \triangle ACB \), the same triangle with the vertices \( B \) and \( C \) ordered differently. By hypothesis

\[
AB = AC \quad \text{and} \quad AC = AB.
\]

Moreover, by the symmetry axiom of angle measurement,

\[
m\angle BAC = m\angle CAB.
\]

By Side–Angle–Side, \( \triangle ABC \) is congruent to \( \triangle ACB \). It follows that corresponding angles are congruent. Hence

\[
m\angle ABC = m\angle ACB. \quad \Box
\]

What about the converse of this theorem? If we know that \( \angle B \) is congruent to \( \angle C \) does it follow that \( AB = AC \), that is, that the triangle is isosceles? The answer is “yes” but the proof requires a different method for proving triangle congruence than Side–Angle–Side.

We discuss four other methods for establishing triangle congruence.

**Side–Side–Side** [SSS] \ Given \( \triangle ABC \) and \( \triangle XYZ \). If \( AB = XY \), \( AC = XZ \), and \( BC = YZ \), then the two triangles are congruent.
SSS is a true statement. It is not an axiom, however, but can be proved using Side–Angle–Side [or SAS], although we will not do so in these notes.

**Angle–Side–Angle** [ASA] Given $\triangle ABC$ and $\triangle XYZ$. If $AB = XY$, $m\angle A = m\angle X$, and $m\angle B = m\angle Y$, then the two triangles are congruent.

**Angle–Angle–Side** Given $\triangle ABC$ and $\triangle XYZ$. If $m\angle A = m\angle X$, $m\angle B = m\angle Y$, and $BC = YZ$, then the two triangles are congruent.

ASA and AAS are both true statements which can be proved using SAS.

Now we come to the tricky one:

**Side–Side–Angle** [SSA] (Warning: Reversing these three letters leads to snickers among seven graders.) Given $\triangle ABC$ and $\triangle XYZ$. If $AB = XY$, $BC = YZ$, and $m\angle C = m\angle Z$, then are the two triangles congruent?

It turns out that SSA is *not* a valid method for establishing triangle congruence. SSA is “almost” true, however, as the following examples show.

**Example 4.7.** Suppose we specify $\angle C$ and the two lengths $BC$ and $AB$.

1. If $m\angle C = 45$, $BC = 2$, and $AB = 1.75$, then two different triangles can be drawn matching these condition. One way $\angle A$ is obtuse, the other way it is acute.
2. No triangle can be drawn with the conditions $m\angle C = 45$, $BC = 2$, and $AB = 1$. The specified length $AB = 1$ is not long enough for side $\overline{AB}$ to reach $\overline{AC}$.
3. Exactly one triangle can be drawn with the conditions $m\angle C = 90$, $BC = 2$, and $AB = 3$. In general SSA is a valid method for triangle congruence if the angle is a right angle. This fact is important enough to have its own name.

**Hypotenuse–Leg** [HL] If the hypotenuse and one leg of a right triangle are congruent to the hypotenuse and leg of a second right triangle, then the two triangles are congruent.

We shall feel free to employ all the methods—SAS, SSS, ASA, AAS, and HL—listed in this section to establish triangle congruence. The important thing for you to realize is that only Side–Angle–Side is an axiom; the others can be proved, though we shall not do them all. As an illustration, we outline a proof of Angle–Side–Angle.

Given $\triangle ABC$ and $\triangle XYZ$. Suppose $\angle A \cong \angle X$, $\angle B \cong \angle Y$, and $AB = XY$. We wish to prove that $\triangle ABC \cong \triangle XYZ$. Compare $BC$ and $YZ$. If $BC = YZ$, then the two triangles
are congruent by SAS, and we’re done. So assume \(BC \neq YZ\). Without loss of generality, we may assume that \(BC > YZ\).

*First Time-out.* What’s all this “without loss of generality” stuff? Math books are always saying annoying things like this and students are generally thinking “whatever.” How can you just assume that \(BC > YZ\)? If \(BC \neq YZ\), isn’t the inequality \(BC < YZ\) just as likely to occur? The answer is that if \(BC\) is shorter than \(YZ\), then we could just switch the letters \(A, B, C\) and \(X, Y, Z\), so that we would have \(BC > YZ\). Can you just switch letters like this? Sure, why not? We started with two general triangles—one we called \(\triangle ABC\), the other we labelled \(\triangle XYZ\). Reversing these names from the outset would not have changed the problem. The statements \(\angle A \cong \angle X\) and \(\angle B \cong \angle Y\) still hold if we switch \(A\) with \(X\) and \(B\) with \(Y\); the same goes for the equation \(AB = XY\). Had there been something different about the two triangles, for instance, if we assumed that one of the triangles was isosceles, but not necessarily the other triangle, then we could not be so free in switching letters.

Back to the proof. Use the Ruler Axiom to construct a point \(D\) on \(\overrightarrow{BC}\) such that \(BD = YZ\). The point \(D\) lies inside \(\overrightarrow{BC}\), otherwise \(D\) would satisfy the betweenness relation \(B - C - D\), implying that \(BC < BD = YZ\), which is false. If you’re thinking “couldn’t \(D = C\)?”, forget it: \(D = C\) leads to \(BD = YZ\), another false statement. The only choice left is \(B - D - C\), proving that \(D\) lies inside \(\overrightarrow{BC}\), as claimed. By SAS, \(\triangle ABD\) is congruent to \(\triangle XYZ\). Matching corresponding parts of congruent triangles, we get the angle congruence \(\angle BAD \cong \angle YXZ\). Switching gears, we argue that since \(B - D - C\), it surely follows that \(AB - AD - AC\).

*Second Time-out.* Not so fast. How do we know that the ordering of the rays \(\overrightarrow{AB}, \overrightarrow{AD}, \) and \(\overrightarrow{AC}\) preserves the ordering of the points \(B, D,\) and \(C\)? The answer is, “We don’t. We really need another axiom to guarantee this.

**Order Compatibility Axiom.** Given a triangle \(\triangle ABC\) and a point \(D\) on line \(\overrightarrow{BC}\). The betweenness relation \(B - D - C\) holds if and only if \(\overrightarrow{AB} - \overrightarrow{AD} - \overrightarrow{AC}\) is true.

If you spotted this problem in the proof, then you should consider a career in either law or mathematics. By the way, if you are thinking to yourself “couldn’t you just *see* the correct betweenness relation from the diagram?”, the answer is that diagrams can be misleading. Section 11 is all about what happens when we rely too much on diagrams to determine betweenness relations. In general, we want our proofs to rely solely on axioms, already established theorems, and logic.
Back to the proof. The betweeness relation $\overrightarrow{AB} - \overrightarrow{AD} - \overrightarrow{AC}$ implies that $m\angle BAD < m\angle BAC$. The first angle $\angle BAD$ is congruent to $\angle YXZ$, as noted above. But the second angle $\angle BAC$ is also congruent to $\angle YXZ$, by hypothesis. So we have proved that $m\angle YXZ = m\angle BAD < m\angle BAC = m\angle YXZ$. This obvious contradiction originated from the assumption that $BC \neq YZ$. We must have been wrong to make this assumption, and in fact, $BC = YZ$, and consequently, $\triangle ABC \cong \triangle XYZ$.

4.9 Exercises

1. Draw a picture illustrating each of the three examples of SSA listed above.

2. Use ASA to prove the converse of the isosceles triangle theorem, that is, that in $\triangle ABC$, if $m\angle B = m\angle C$, then $AB = AC$.

3. Prove: If the hypotenuse and one acute angle of a right triangle are congruent to the hypotenuse and one acute angle of a second right triangle, then the two triangles are congruent.

4. Use the Order Compatability Axiom to prove: given triangle $\triangle ABC$ and point $D$ on ray $\overrightarrow{BC}$, we have that $BD > BC$ if and only if $m\angle BAD > m\angle BAC$. You can test this theorem with a pair of scissors and a ruler.

10. Perpendiculars

**Definition 4.7.** Two angles $\angle A$ and $\angle B$ are called **supplementary** if and only if

$$m\angle A + m\angle B = 180.$$  

**Theorem 4.11** (Supplementary Angle Theorem). Let $h$ and $k$ be opposite rays with a common endpoint $A$. Suppose $r$ is a ray with endpoint $A$ other than $h$ or $k$. Then $h - r - k$. Consequently, the angle formed by the rays $h$ and $r$ and the angle formed by the rays $r$ and $k$ are supplementary.

**Proof.** Let $\ell$ be the line $h \cup k$. Since $h$, $r$, and $k$ lie on the same halfplane determined by the line $\ell$, the Betweenness Axiom for Rays guarantees that a betweenness relation

$h - r - k$, $h - k - r$, or $r - h - k$
holds. Since \( m(h, k) = 180 \) by Angle Measure Axiom 3, the only possible betweenness relation among the three listed above is

\[ h - r - k. \]

(The second choice \( h - k - r \) implies \( m(h, r) = m(h, k) + m(k, r) = 180 + m(k, r) > 180 \), a contradiction; similarly, the third choice \( r - h - k \) implies \( m(r, k) > 180 \).) By definition of betweenness of rays, we have \( m(h, r) + m(r, k) = m(h, k) = 180 \). Hence the angles formed by \( h, r \) and \( r, k \) are supplementary.

An immediate consequence of the definition of supplementary angles is

**Theorem 4.12** (Congruent Supplementary Angle Theorem). Let \( \angle A \) and \( \angle B \) be congruent supplementary angles. Then \( \angle A \) and \( \angle B \) are both right angles.

**Do–It–Now Exercise.** Find a (really) easy way to prove this theorem.

**Definition 4.8.** Two lines \( \ell \) and \( m \) are called **parallel** if and only if they do not intersect.

Two non-parallel lines meet in exactly one point, because of Incidence Axiom 3 which asserts that exactly one line passes through two distinct points. At the point of intersection of non-parallel lines \( \ell \) and \( m \), we can form four angles:

\[
\begin{array}{c}
\angle 1 \\
\angle 2 \\
\angle 3 \\
\angle 4
\end{array}
\]

**Definition 4.9.** In the above diagram, the pair of angles \( \angle 1 \) and \( \angle 4 \) as well as the pair \( \angle 2 \) and \( \angle 3 \) are called **vertical** angles.

By the Supplementary Angle Theorem,

\[ m\angle 1 + m\angle 2 = 180 \]
and
\[ m\angle 1 + m\angle 3 = 180. \]
It follows that
\[ m\angle 2 = m\angle 3. \]
A similar argument proves that
\[ m\angle 1 = m\angle 4. \]
This proves

**Theorem 4.13 (Vertical Angle Theorem).**

When two lines cross, their vertical angles are congruent.

**Definition 4.10.** We say that two non-parallel lines \( \ell \) and \( m \) are **perpendicular** if in the intersection diagram \( m\angle 1 = 90 \).

If follows from the Supplementary and Vertical Angles Theorems that when \( \ell \) and \( m \) are perpendicular, then all four angles are right angles. The Protractor Axiom guarantees that we can construct a perpendicular line at any point \( A \) on a given line. It is worth stating this as a theorem.

**Theorem 4.14.** Given any point \( A \) on a line \( \ell \).

There is a line through \( A \) that is perpendicular to \( \ell \).

Now suppose we are given a line \( \ell \) and a point \( A \) not on \( \ell \). Is it always the case that we can construct a line through \( A \) that is perpendicular to \( \ell \)? The answer is “yes” and here’s how.

Pick any point \( B \) on line \( \ell \). Now pick a second point \( C \) on \( \ell \) so that \( m\angle ABC \leq 90 \). How do we know we can always find such a point \( C \)? We begin by choosing \( D \) to be any point other than \( B \) on \( \ell \). If \( m\angle ABD \leq 90 \), fine; take \( C = D \). If \( m\angle ABD > 90 \), then choose a point \( C \) on \( \ell \) so that \( \overrightarrow{BC} \) and \( \overrightarrow{BD} \) are opposite rays. Then \( \angle ABC \) and \( \angle ABD \) are supplementary angles. Hence, \( m\angle ABC \leq 90 \).

Now if \( m\angle ABC \) is exactly 90, stop, we’re done; line \( \overrightarrow{AB} \) is perpendicular to line \( \overrightarrow{BC} \).

We are left with the case where \( m\angle ABC < 90 \). On the side of line \( \ell \) which does not contain \( A \) construct a ray \( h \) with endpoint \( B \) such that the angle measure between \( \overrightarrow{BC} \) and ray \( h \) is precisely \( m\angle ABC \). Using the Ruler Axiom, construct a point \( E \) on ray \( h \) so that \( BE = BA \).
Observe that $A$ and $E$ are on opposite sides of $\ell$, so the line segment $\overline{AE}$ must cross line $\ell$ at a point $M$. Examine the two triangles $\triangle ABM$ and $\triangleEBM$. By construction

$$m\angle ABM = m\angle ABC = m\angle EBC = m\angle EBM.$$  

Furthermore,

$$AB = EB$$

and finally, the two triangles share the common side $\overline{BM}$. By axiom Side–Angle–Side,

$$\triangle ABM \cong \triangleEBM.$$  

This means

$$\angle AMB \cong \angle EMB.$$  

Since $A$, $M$, and $E$ are collinear, the angles $\angle AMB$ and $\angle EMB$ are supplementary angles. But we know that congruent, supplementary angles must be right angles. We have found our perpendicular line: $\overline{AE}$ is perpendicular to $\ell$.

We have just proven the following statement:

**Theorem 4.15.** Given a line $\ell$ and a point $A$ not on $\ell$.

Then there is a line $t$ through $A$ and perpendicular to $\ell$.

The line $t$ guaranteed by Theorem 4.15 crosses the original line $\ell$ at a point $B$. We define the distance from $A$ to line $\ell$ to be the distance $AB$. 


4.10 Exercises

1. Suppose two lines $\ell$ and $m$ intersect at point $A$. We say that the point $P$ is **equidistant** from lines $\ell$ and $m$ if the (perpendicular) distance from $A$ to $\ell$ equals the (perpendicular) distance from $A$ to $m$. Describe the sets of all points $P$ which are equidistant from $\ell$ and $m$.

2. Given: $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{AD}$
   $\overrightarrow{BC}$ is perpendicular to $\overrightarrow{CD}.$
   $AB = BC$

   Prove that $AD = DC$.

3. Given $\triangle ABC$. Prove: $\angle B + \angle C < 180$.

   Outline of the proof:
   Let $M$ be the midpoint of $\overline{AB}$.
   Let $E$ be a point on the ray $\overrightarrow{CM}$ such that $CE = 2CM$.
   Show that $\triangle AMC \cong \triangle BME$.
   Show that $\overrightarrow{BC} - \overrightarrow{BM} - \overrightarrow{BE}$. Hint: Use the Order Compatibility Axiom.
   Show that $\angle A + \angle B = \angle CBE$.
   Since $\angle CBE$ is a proper angle, conclude that $\angle A + \angle B < 180$.

4. This exercise asks you to prove

   **Theorem 4.16** (Exterior Angle Theorem). Given $\triangle ABC$ and $A - C - D$. Then $\angle ABD > \angle A$ and $\angle ABD > \angle B$.

   Angle $\angle ABD$ is called an **exterior angle** of triangle $\triangle ABC$. Angles $\angle A$ and $\angle B$ are called the **remote interior angles** corresponding to $\angle ABD$.

   ![Diagram](image)

   Hint: Use Exercise 3 and the Supplementary Angle Theorem.
11. Isosceles Triangle Paradox

Consider the following “proof” that all triangles are isosceles.

1. Let $M$ be the midpoint of $AB$. Draw the bisector of $\angle C$ and the perpendicular bisector of $AB$. Let $E$ be their point of intersection.

2. By construction, $EM \perp AB$. From point $E$ drop perpendiculars $EF$ onto $AC$ and $EG$ onto $BC$.

3. $\triangle CFE \cong \triangle CGE$ by Angle–Angle–Side. Each is a right triangle with $CE$ as a hypotenuse and $\angle FCE = \angle GCE$ since $CE$ bisects $\angle C$.

4. $EF = EG$ and $CF = CG$ [congruent parts of the congruent triangles $\triangle CFE$ and $\triangle CGE$].

5. $\triangle AEM \cong \triangle BEM$ by Side–Angle–Side. Both are right triangles, $AM = BM$ since $M$ is the midpoint of $AB$, and they share side $EM$.

6. $AE = BE$ [CPCT].
7. \( \Delta AEF \cong \Delta BEG \) by Hypotenuse-Leg. Both are right triangles, \( EF = EG \) from step 4, and \( AE = BE \) from step 6.

8. \( FA = GB \) [CPCT]

9. Combining steps 4 and 8, we have
\[
CA = CF + FA = CG + GB = CB.
\]

10. Since \( CA = CB \) it follows that \( \Delta ABC \) is isosceles.

Something is very wrong here! We have just “proved” a statement that is categorically false. Clearly not all triangles are isosceles. Try to find the flaw in the proof. It may help you to draw a triangle which is visually not isosceles, such as a 3–5–7 triangle. Construct the angle bisector of \( \angle C \) using a compass or protractor. Construct the perpendicular bisector of \( \overline{AB} \) using a compass or ruler. Where do these lines intersect? Do you see why the concept of betweenness is so important?

12. Parallels

Given one rail of a railroad track, is there always a second rail whose (perpendicular) distance from the first rail is exactly the width across the tires of a train, so that the two rails never intersect? Of course we believe this is so. Let’s state it as a theorem.

**Theorem 4.17.** Given a line \( \ell \) and a point \( P \) not on \( \ell \).

Then there exists a line through \( P \) parallel to \( \ell \).

Okay, how do we go about proving Theorem 4.17? One idea is to use perpendiculars. Construct a line \( t \) through \( P \) and perpendicular to \( \ell \). Now construct a line \( m \) through \( P \) and perpendicular to \( t \). Surely, lines \( m \) and \( \ell \) must be parallel. Suppose you are driving on 8th Avenue in New York City. You turn right on 34th Street, go two blocks, and then turn right on 6th Avenue. Sixth Avenue must be parallel to 8th Avenue—and it is. But can we prove it?

Here goes. Suppose, to play Devil’s Advocate, that \( m \) and \( \ell \) are not parallel. Then they must meet at a point which we will call \( X \) (the “mysterious” point). Let \( Q \) be the point of intersection of line \( t \) and \( \ell \). Note that \( \angle QPX \) and \( \angle PQX \) are both right angles. Now let \( Y \) be a point on the opposite ray of \( \overline{QX} \) so that \( QY = QX \).
Consider the triangles $\triangle PQY$ and $\triangle PQX$. Since the angles $\angle PQY$ and $\angle PQX$ are supplementary and $m\angle PQX = 90$, it follows that $m\angle PQY = 90$. By construction $QY = QX$ and finally the two triangles share the side $\overline{PQ}$. By Side–Angle–Side,

$$\triangle PQY \cong \triangle PQX.$$ Consequently,

$$m\angle QPY = m\angle QPX = 90.$$ By angle addition,

$$m\angle YPX = m\angle YPQ + m\angle QPX = 90 + 90 = 180.$$ In other words, $\angle YPX$ is a straight angle, implying that $Y$, $P$, and $X$ are collinear. We now have two different lines $m$ and $\ell$ passing through the points $X$ and $Y$. That ain’t right! One of our basic axioms states that only one line passes through two distinct points. We were led to this contradiction by assuming that $m$ and $\ell$ intersect; so, in fact, $m$ and $\ell$ must be parallel. \qed

Now here comes the question that mystified mathematicians for over two thousand years:

*Is $m$ the only parallel line through $P$?*

It certainly seems plausible that any other line, different than the line constructed by perpendiculars in the previous theorem, must meet $\ell$.

**Theorem 4.18.** [The Parallel Postulate] Given a line $\ell$ and a point $P$ not on $\ell$.

Then there exists a *unique* line through $P$ parallel to $\ell$.

A line $t$ which crosses two given lines $\ell$ and $m$ is called a **transversal**. In general, eight angles are formed:
Angles formed between the lines $\ell$ and $m$ are called **interior** angles. In the above diagram, angles $\angle 1$, $\angle 2$, $\angle 5$, and $\angle 6$ are all interior angles. Note that interior angles come in pairs: $\angle 1$ and $\angle 5$ both lie on the left side of $t$, while $\angle 2$ and $\angle 6$ both lie on the right side of $t$.

**Euclid’s Fifth Postulate:** In a transversal configuration, if the sum of the measure of the two interior angles on the same side of the transversal $t$ is less than 180, then the lines $\ell$ and $m$ will eventually intersect on this particular side of $t$.

This axiom seems self evident. Suppose $m \angle 2 + m \angle 6 < 180$. Then the two lines $\ell$ and $m$ in the diagram above are “tilted” towards each other on the right side of $t$. If they are extended far enough they will eventually meet on the right side of $t$.

Euclid’s Fifth Postulate shows that there cannot be more than one parallel line through a given point $P$. If the sum $m \angle 2 + m \angle 6$ is less than 180 then the two lines will meet on the side of $t$ that contains $\angle 2$ and $\angle 6$; otherwise $m \angle 1 + m \angle 5 < 180$ and the two lines meet on the other side of $t$. Lines $\ell$ and $m$ are parallel if and only if $m \angle 2 + m \angle 6$ is exactly 180. Interior angles which lie on opposite halfplanes of $t$ are called **alternate interior** angles. In the diagram, the pair $\angle 1$ and $\angle 6$ are alternate interior angles. So are the pair $\angle 2$ and $\angle 5$.

**Do–It–Now Exercise.** Using Euclid’s Fifth Postulate, prove

**Theorem 4.19** (Alternate Interior Angles Theorem). Given a configuration of two lines $\ell$ and $m$ cut by a transversal $t$.

The lines $\ell$ and $m$ are parallel if and only if alternate interior angles are congruent.
Do–It–Now Exercise. Show that Theorem 4.18, the Parallel Postulate, is an easy consequence of Theorem 4.19.

At this point it seems fair to ask: What is the status of Euclid’s Fifth Postulate? Do we add it to our system of axioms or is it possible to prove it from the other axioms of geometry? For centuries mathematicians attempted to prove Euclid’s Fifth Postulate (or equivalently, the Parallel Postulate) from the other axioms of Euclid. Finally it was realized that the search for the elusive proof was in vain. Geometries in which all the other axioms of Euclid hold except for the Parallel Postulate were discovered (invented?) which are as consistent as Euclidean geometry in which the Parallel Postulate is assumed to hold true.

Our plan is to stay tuned to the Euclid channel for now, accept Euclid’s Fifth Postulate and see where it leads us. Later in Module 7 we will explore the “non-Euclidean” channels.

4.12 Exercises

1. Given an isosceles triangle $\triangle ABC$, with $AB = AC$. Suppose $\ell$ is a line parallel to $\overline{AB}$ and that $\ell$ meets segment $\overline{AC}$ at point $P$ and $\ell$ meets segment $\overline{BC}$ at point $Q$. Prove that $\triangle PCQ$ is isosceles.

2. Given: $\overline{AB}$ is parallel to $\overline{DE}$

   \[
   AB = DE
   \]

   \[
   BE = CF
   \]

   Prove: $\overline{AC}$ is parallel to $\overline{DF}$.

3. One way of constructing a line through a point $P$ parallel to a given line $\ell$ is to use the Alternate Interior Angles Theorem. Here is another way.

   Given a line $\ell$ and a point $P$ not on $\ell$.

   Let $A$ be any point on $\ell$.

   Locate the midpoint $M$ of $\overline{AP}$.

   Draw any line through $M$ which meets $\ell$ at a point $C$ other than $A$.

   Find the point $D$ on $\overline{CM}$ such that $CD = 2CM$.

   Show that $\overline{BP}$ is parallel to line $\ell$. 
13. Angle Sums of Triangles

We are now ready to prove the most famous of all theorems in geometry.

**Theorem 4.20.** In $\triangle ABC$

$$m\angle A + m\angle B + m\angle C = 180.$$ 

The idea of the proof is to use the line $m$ through $A$ parallel to $BC$.

By the Alternate Interior Angle Theorem,

$$m\angle A = m\angle 1$$

and

$$m\angle B = m\angle 2.$$ 

Thus

$$180 = m\angle 1 + m\angle C + m\angle 2$$ 

$$= m\angle A + m\angle C + m\angle B.$$ 

We’re done. \hfill \square 

**Definition 4.11.** Two angles $\angle A$ and $\angle B$ are said to be **complementary** if

$$m\angle A + m\angle B = 90.$$ 

**Theorem 4.21.** In triangle $\triangle ABC$, if $\angle C$ is a right angle, then $\angle A$ and $\angle B$ are complementary.
4.13 Exercises

1. If $m\angle A = 40$ and $m\angle B = 70$, what is $m\angle C$ of $\triangle ABC$?

2. If $m\angle A = 2m\angle B$ and $m\angle C$ of $\triangle ABC$ is 120, what is $m\angle A$ and $m\angle B$?

3. If one angle of an isosceles triangle has measure 45, what are the measures of the other two angles?

4. What is the measure of an angle of an equilateral triangle?

5. Prove that the measure of the short leg of a 30–60–90 triangle is half the measure of the hypotenuse.


7. Given that lines $\ell$ and $m$ are parallel. Find the measures of the other 12 angles in the following diagram:

8. [The Star Problem] Given a five pointed star:
Prove that $m\angle A + m\angle B + m\angle C + m\angle D + m\angle E = 180$.

9. Given: $B - C - D$

- $\angle ABC$ is a right angle
- $\angle CDE$ is a right angle
- $AB = CD$
- $BC = DE$

Prove: $\angle ACE$ is a right angle.

In the Euclidean model of the $x-y$ plane, suppose the points $A - E$ are given by

$A = (0, b), \ B = (0, 0), \ C = (a, 0), \ D = (a+b, 0), \ E = (a+b, a)$

(a) What is the slope of line $\overrightarrow{AC}$?
(b) What is the slope of line $\overrightarrow{CE}$?
(c) What conclusion can you make about the slopes of perpendicular lines?