6. Induction

**Principle of Induction.** Let \( P(n) \) be a statement about the positive integer \( n \). In order to show that \( P(n) \) is true for all positive integers \( n \), it suffices to show that

(i) **First Case:** \( P(1) \) is true;

(ii) **Next Case:** If \( P(n) \) is true for the integer \( n \), then statement \( P \) is true for the next integer \( n + 1 \).

**Example:** Find the sum of consecutive odd numbers. Let’s experiment:

\[
\begin{align*}
1 &= 1 \\
1 + 3 &= 4 \\
1 + 3 + 5 &= 9 \\
1 + 3 + 5 + 7 &= 16 \\
\end{align*}
\]

Question: What is the pattern 1, 4, 9, 16?

Answer: They are all squares.

Question: Does this pattern continue for the next sum when \( n = 5 \)?

Answer: Let’s check:

\[
1 + 3 + 5 + 7 + 9 = 16 + 9 = 25,
\]

another square.

**Conjectured Formula:**

\[
1 + 3 + 5 + \cdots + (2n - 1) = n^2
\]

**Geometric Proof:** For example, when \( n = 5 \), we can decompose a 5 square into five L-shaped pieces whose areas are 1, 3, 5, 7, and 9.

\[
\begin{array}{c}
1 \\
\vdots \\
9
\end{array}
\]

**Induction Proof:**
$P(n) : 1 + 3 + 5 + \cdots + (2n - 1) = n^2$

First Step: $P(1)$ is true: $1 = 1$

Next Step: Assume $P(n)$ is true and show that $P(n+1)$ is also true. The idea is to start with the formula $P(n)$. Add the next odd number $2(n+1) - 1$ to both sides, and hopefully transform the equation into $P(n+1)$, like this:

\[
\begin{align*}
1 + 3 + 5 + \cdots + (2n - 1) &= n^2 \quad [P(n)] \\
1 + 3 + 5 + \cdots + (2n - 1) + (2(n+1) - 1) &= n^2 + 2(n+1) - 1 \quad \text{[add } 2(n+1) - 1]\ \\
1 + 3 + 5 + \cdots + (2n - 1) + (2(n+1) - 1) &= n^2 + 2n + 1 \quad \text{[algebra]} \\
1 + 3 + 5 + \cdots + (2n - 1) + (2(n+1) - 1) &= (n + 1)^2 \quad \text{[factoring]}.
\end{align*}
\]

This last equation is precisely $P(n+1)$, the next case of the proposition we want to prove. By induction, $P(n)$ is true for all positive integers. \qed

Exercise #19. Find a formula for the sum

\[T = 1 + 2 + 3 + \cdots + n\]

These numbers are called triangular numbers. Do you see why?

Exercise #20. Find a formula for the sum of the squares

\[S = 1^2 + 2^2 + 3^2 + \cdots + n^2\]

The following table lists the values of $n$, $S$, $3S$, and a factorization of $3S$.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$S$</th>
<th>$3S$</th>
<th>factors</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1·3</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>15</td>
<td>3·5</td>
</tr>
<tr>
<td>3</td>
<td>14</td>
<td>42</td>
<td>6·7</td>
</tr>
<tr>
<td>4</td>
<td>30</td>
<td>90</td>
<td>10·9</td>
</tr>
<tr>
<td>5</td>
<td>55</td>
<td>165</td>
<td>15·11</td>
</tr>
<tr>
<td>6</td>
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<td>7</td>
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<tr>
<td>8</td>
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</tr>
<tr>
<td>9</td>
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<td></td>
</tr>
<tr>
<td>10</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(1) Complete the table for the values $n = 6$ to 10.
(2) Guess a formula for $S$.
(3) Prove your formula by induction.
Now use induction to prove the following theorem about congruences:

**Theorem #20.** Let $a$, $b$, $k$, and $n > 0$ be integers. If $a \equiv b \pmod{n}$, then $a^k \equiv b^k \pmod{n}$.

One interesting application of this theorem is that when working mod 9, all powers of 10 are congruent to one. That is, $10 \equiv 1 \pmod{9}$ implies $10^k \equiv 1^k = 1 \pmod{9}$ for all exponents $k$. Use this fact to prove that the following divisibility test:

**Theorem #21.** The number $n$ is divisible by 9 if and only if the sum of the base 10 digits of $n$ is divisible by 9.

**Problem #22.** For each divisor $d = 2, 3, 4, \cdots, 11, 12$, devise a test to determine whether the number $n$ written in decimal as 

$$n = d_t d_{t-1} \cdots d_3 d_2 d_1 d_0$$

is divisible by $d$. 