MODULE 4

Mediants, Contiguous Numbers, and Farey’s Sequence

1. Victoria’s Theorem

Victoria was asked to find a fraction between \( \frac{2}{5} \) and \( \frac{3}{7} \).

**Fifth Grade Textbook Solution:** Find the midpoint of these two fractions:

\[
P = \frac{1}{2} \left( \frac{2}{5} + \frac{3}{7} \right)
\]

\[
= \frac{1}{2} \left( \frac{14}{35} + \frac{15}{35} \right)
\]

\[
= \frac{1}{2} \cdot \frac{29}{35}
\]

\[
= \frac{29}{70}
\]

**Victoria’s Dad’s Solution:** Add the numerator and denominator to get

\[
M = \frac{2 + 3}{5 + 7} = \frac{5}{12}
\]

Is this answer correct?

A little help from a calculator shows that

\[
\frac{2}{5} = .40000 \cdots < \frac{5}{12} = .41666 \cdots < \frac{3}{7} = .42857 \cdots
\]

When her teacher marked her answer incorrect (because it was not the answer she expected) her father contacted the teacher and explained that since there is more than one fraction between \( \frac{2}{5} \) and \( \frac{3}{7} \), there is more than just one answer; with \( \frac{5}{12} \) being correct.

It turns out that this method always works and that, in some sense, it is the very best possible answer.
Exercise #79. Show that if \( \frac{a}{b} < \frac{c}{d} \) are two positive fractions, then

\[
\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}.
\]

**Definition 4.1.** If \( \frac{a}{b} \) and \( \frac{c}{d} \in \mathbb{Q} \), then \( \frac{a+c}{b+d} \) is called their mediant. We will write

\[
\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}.
\]

This seems a bit naughty—it looks like we are adding fractions by the **totally wrong** way of adding numerator and denominator. The point is that \( \frac{a}{b} \oplus \frac{c}{d} \) is not the sum of the fractions; it is the mediant. We must be careful never to confuse \( \oplus \) with +.

### 2. Contiguous Fractions

In working with fractions we shall always abide by the following two conventions:

(i) we will assume that a given fraction \( \frac{a}{b} \) is reduced, that is, \( \gcd(a, b) = 1 \), and

(ii) if a fraction \( \frac{a}{b} \) is negative, then we will assume that the numerator \( a \) is negative, that is, we will think of \( -\frac{3}{7} \) as \( -\frac{3}{7} \) and not as \( \frac{3}{-7} \).

**Definition 4.2.** Let \( S \) be a set of reduced rational numbers with distinct denominators greater than zero. Then the element of \( S \) with least denominator is said to be the **simplest** element of \( S \).

**Example:** \( S = \left\{ \frac{109}{83}, \frac{2}{91}, \frac{108}{1237}, \frac{23107}{30} \right\} \) has simplest element \( \frac{23107}{30} \).

**Definition 4.3.** Two fractions \( \frac{a}{b} < \frac{c}{d} \in \mathbb{Q} \) which satisfy \( ad + 1 = bc \) are called **contiguous**.

Note that \( \frac{a}{b} + \frac{1}{bd} = \frac{c}{d} \)

**Exercise #80.** Show that the fractions \( \frac{a}{b} = \frac{2}{5} \) and \( \frac{c}{d} = \frac{3}{7} \) are contiguous.

**Discussion Problem #81.** How are the fractions \( \frac{a}{b} = \frac{2}{5} \) and \( \frac{c}{d} = \frac{3}{7} \) related to the Pails of Water Problem for 5 and 7?

**Theorem #82.** Suppose \( \frac{a}{b} \) and \( \frac{c}{d} \) are contiguous. Then any rational number \( \frac{p}{q} \) satisfying \( \frac{a}{b} < \frac{p}{q} < \frac{c}{d} \) has denominator \( q \) greater than \( \max(b, d) \).
Exercise #83. If \( \frac{a}{b} = \frac{2}{5} \) and \( \frac{c}{d} = \frac{3}{7} \), show that the mediant \( \frac{a}{b} \oplus \frac{c}{d} \) is contiguous to both \( \frac{a}{b} \) and \( \frac{c}{d} \).

It turns out that this behavior is not an accident. It is always the case that the mediant of two contiguous fractions is contiguous to both of them.

Theorem #84. Suppose \( \frac{a}{b} < \frac{c}{d} \) are contiguous. Then the simplest rational number between them is \( \frac{a+c}{b+d} \). Moreover, \( \frac{a+c}{b+d} \) is contiguous to both \( \frac{a}{b} \) and \( \frac{c}{d} \).

3. The Farey Sequence

How do you go about finding contiguous numbers, say in \([0, 1]\)?

Constructive Method: Filtered by the size of the denominators.

Note that \( \frac{0}{1} \) and \( \frac{1}{1} \) are contiguous: \( \frac{0}{1} + \frac{1}{1} = \frac{1}{1} \)

Start with the set \( F_1 = \left\{ \frac{0}{1}, \frac{1}{1} \right\} \).

What is the simplest number between them?

Answer: By the previous Theorem, it is \( \frac{0+1}{1+1} = \frac{1}{2} \).

Put \( F_2 = \left\{ \frac{0}{1}, \frac{1}{2}, \frac{1}{1} \right\} \).

Now \( \frac{1}{2} \) is contiguous, with numbers around it.

So \( F_3 = \left\{ \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\} \).

Exercise #85. Find \( F_4, F_5, F_6, F_7, F_8, F_9, F_{10} \).

Definition 4.4. \( F_n \) is called the Farey sequence of order \( n \).

It is formed by taking all mediants between successive elements of \( F_{n-1} \) so long as the resulting denominator is \( \leq n \).

Theorem \( F_n \) consists of the sequence of fractions \( \frac{a}{b} \in [0, 1], b \leq n, (a, b) = 1 \), arranged in increasing order.
Exercise #86. Find the fraction $\frac{a}{b}$ immediately before and the fraction $\frac{c}{d}$ immediately after $\frac{61}{79}$ in the Farey sequence $F_{100}$.

[Hint: Pails of Water]

Exercise #87. Let $\frac{a}{b}$ and $\frac{c}{d}$ be the fractions immediately to the left and right of the fraction $\frac{1}{2}$ in $F_n$. Prove that

$$b = d = 1 + 2\left\lfloor \frac{n-1}{2} \right\rfloor,$$

where $\lfloor x \rfloor$ (the floor function) is the greatest integer less than or equal to $x$. ($\lfloor \pi \rfloor = 3$, $\lfloor \sqrt{20} \rfloor = 4$) Also, prove that $a + c = b$.

Exercise #88. Let $\frac{a}{b}$ and $\frac{c}{d}$ run through all pairs of adjacent fractions in $F_n$, $n > 1$. Prove that

$$\min(\frac{c}{d} - \frac{a}{b}) = \frac{1}{n(n-1)} \quad \text{and} \quad \max(\frac{c}{d} - \frac{a}{b}) = \frac{1}{n}.$$

Definition 4.5. The infinite Farey sequence of order $n$ is the sequence of all reduced fractions with denominators $\leq n$ listed in order of size. Notation: $F_n$

Example. For $n = 3$, $F_3$ is

$$\cdots \frac{2}{1}, \frac{-4}{3}, \frac{-1}{2}, \frac{-1}{3}, 0, 1, 1, 2, 1, 3, 4, 5, 2, \frac{1}{3}, \frac{3}{2}, \frac{3}{3}, \frac{2}{3}, \frac{1}{1}, \cdots$$

Question #89. Explain how this sequence can be obtained from the Farey Sequence $F_3$.

Question #90. How do we know the consecutive terms are contiguous?

Theorem #91. If $\frac{a}{b}$, $\frac{c}{d}$ $\in F_n$ and are consecutive, then

$$\left| \frac{a}{b} - \frac{a+c}{b+d} \right| = \frac{1}{b(b+d)} \leq \frac{1}{b(n+1)}.$$

Theorem #92. If $n \in \mathbb{N}$, $x \in \mathbb{R}$, then there exists $\frac{a}{b} \in \mathbb{Q}$ such that $0 < b \leq n$ and

$$\left| x - \frac{a}{b} \right| \leq \frac{1}{b(n+1)}.$$

Exercise #93. Suppose in the previous theorem that $x = \pi = 3.1415926\cdots$ and $n = 10$, what fraction $\frac{a}{b}$ satisfies the inequality $|x - \frac{a}{b}| \leq \frac{1}{b(n+1)}$?