Clock Arithmetic

It’s 6 o’clock at night. You eat supper for one hour, watch TV for 2 hours, study for 4 hours. What time are you done studying? If we add the time intervals, we get $6 + 1 + 2 + 4 = 13$. But 13 o’clock is not a valid time (except in the army). The time will be 1 in the morning, which we get by subtracting 12 from 13. In mathematics we write this as:

$$13 \text{ mod } 12 = 1.$$ 

On a standard 12-clock, once you reach the number 12, the time resets to 0. Similarly, $25 \text{ mod } 12 = 1$ and $-2 \text{ mod } 12 = 10$, that is, 2 hours before 12 is 10 o’clock.

We are familiar with different clocks besides the 12-clock. For example, we can think of the days of the week as lying on a 7-clock. It is convenient to assign the days the values: Sun = 0, Mon = 1, Tues = 2, · · · , Sat = 6. Suppose you begin a trip for Florida on Wednesday. You spend 2 days driving. You stay in a hotel in Orlando for 6 days, and a hotel in Miami for 5 days. It takes 2 days to drive home. On what day do you return home? Since Wednesday is day number 3, we can add up all the days to get: $3 + 2 + 6 + 5 + 2 = 18$. On a 7-clock, 18 is the same as 4, which we write as:

$$18 \text{ mod } 7 = 4.$$ 

So you return on day number 4, or Thursday.

The number of pennies needed in the exact change for a purchase can be computed using a 5-clock. For example, suppose you buy a $1.99 sandwich, a $.89 drink, and the tax is 26 cents. How many pennies do you need to pay the bill? Adding the three numbers as cents, we have

$$(199 + 89 + 26) \text{ mod } 5 = 314 \text{ mod } 5 = 4.$$ 

You need 4 pennies (plus a dime and 3 dollars).

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In general, on an \( n \)-clock we write

\[ a \mod n = r \]

if \( r \) is the remainder when you divide \( a \) by \( n \). We also say “\( a \) reduces to \( r \mod n \).”

Here are the addition and multiplication tables for 5-clock arithmetic:

\[
\begin{array}{c|cccccc}
+ & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 1 & 2 & 3 & 4 \\
1 & 1 & 2 & 3 & 4 & 0 \\
2 & 2 & 3 & 4 & 0 & 1 \\
3 & 3 & 4 & 0 & 1 & 2 \\
4 & 4 & 0 & 1 & 2 & 3 \\
\end{array}
\quad
\begin{array}{c|cccccc}
\times & 0 & 1 & 2 & 3 & 4 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 2 & 3 & 4 \\
2 & 1 & 4 & 3 & 2 & 1 \\
3 & 4 & 3 & 2 & 1 & 0 \\
4 & 3 & 2 & 1 & 0 & 1 \\
\end{array}
\]

Let’s examine the times table. The zero row and zero column consists of all 0’s. What did we expect? Zero times anything is zero. If we ignore the 0 row and column, the rest of the times table has some interesting properties. Notice, for example, the numbers 1, 2, 3, and 4 are scrambled when we multiply by 2, 3, or 4. That is, each of the rows list the numbers 1, 2, 3, 4 in some order. In the second row we get 2, 4, 1, 3; in the third row we get 3, 1, 4, 2; in the last row the numbers are backwards 4, 3, 2, 1. This scrambling phenomenon is the key idea in constructing the secret codes discussed in the next section.
Exercises:

1. Your birthday is on Monday this year. On what day will your birthday occur next year? Note that the answer depends on whether there is a leap year (giving an additional calendar day February 29) in the upcoming year.

2. Compute

\[(a) \ (11 + 23 - 5) \mod 7\]
\[(b) \ 1000 \mod 7\]
\[(c) \ -250 \mod 33\]

3. To compute \((100 \times 50 + 47) \mod 11\), we first compute \(100 \times 50 + 47 = 5047\) and then find the remainder of 5047 when divided by 11. Since \(5047 = 11 \times 458 + 9\), we find \((100 \times 50 + 47) \mod 11 = 9\). Now is it possible to compute the remainders of the numbers 100, 50, and 47 first, and then use these in the calculation? Since \(9 \times 11 = 99\), it is easy to see that \(100 \mod 11 = 1\). Similarly, \(4 \times 11 = 44\), so \(50 \mod 11 = 6\) and \(47 \mod 11 = 3\). If we compute \((100 \times 50 + 47) \mod 11\) by using the remainders for 100, 50, and 47, we get \((1 \times 6 + 3) \mod 11 = 9\), the correct answer. What rules does this suggest about clock arithmetic?

4. Construct the + and \(\times\) tables for the (a) 7-clock; (b) 6-clock.

5. Give the 7th row of the multiplication table for a 31-clock.