Stirling’s Formula

Our goal here is to prove the following estimate.

**Theorem** (Stirling’s Formula): Fix a $0 < \delta < \pi$ and set $R(\delta) = \{re^{i\theta} : -\pi + \delta \leq \theta \leq \pi - \delta\}$. Then

$$\Gamma(s) = \sqrt{2\pi e^{-s}s^{s-1/2}}(1 + O(1/|s|))$$

for all $s \in R(\delta)$, where the implicit constant depends only on $\delta$.

**Lemma 1:** Suppose $f(x)$ is defined for $x \geq 0$ and $f$ has a continuous derivative. Define $P : \mathbb{R} \to \mathbb{R}$ by $P(x + n) = P(x) = x - 1/2$ for $x \in [0, 1)$ and $n \in \mathbb{Z}$. Then for all positive integers $n$

$$\sum_{i=1}^{n} f(i) - \int_{0}^{n} f(y) dy = \frac{f(0) + f(n)}{2} + \int_{0}^{n} f'(y)P(y) dy.$$

Proof: Fix an integer $i \geq 1$ for the moment. Via integration by parts with $u = y - i - 1/2$ and $dv = f'(y)dy$, we have

$$\int_{i-1}^{i} f'(y)P(y) dy = \int_{i-1}^{i} f'(y)(y - i + 1/2) dy$$

$$= f(y)(y - i + 1/2) \bigg|_{y=i-1}^{i} - \int_{i-1}^{i} f(y) dy$$

$$= \frac{f(i) + f(i - 1)}{2} - \int_{i-1}^{i} f(y) dy.$$

Thus

$$\int_{0}^{n} f'(y)P(y) dy = \sum_{i=1}^{n} \int_{i-1}^{i} f'(y)P(y) dy$$

$$= \sum_{i=1}^{n} \left( \frac{f(i) + f(i - 1)}{2} - \int_{i-1}^{i} f(y) dy \right)$$

$$= \sum_{i=0}^{n} f(i) - \frac{f(0) + f(n)}{2} - \int_{0}^{n} f(y) dy.$$

Proof of Theorem: For $s \in R(\delta)$ write $\log(s) = \log(s) + i\text{Arg}(s)$ with $-\pi < \text{Arg}(s) < \pi$. Set $f(x) = \log(s + x)$ and note that $(s + x)\log(s + x) - x$ is an anti-derivative of $f(x)$. By the Lemma

$$\sum_{i=0}^{n} \log(s + i) = \int_{0}^{n} f(y) dy + \frac{f(0) + f(n)}{2} + \int_{0}^{n} f'(y)P(y) dy$$

$$= (s + y)\log(s + y) - y \bigg|_{y=0}^{n} + \frac{\log(s) + \log(s + n)}{2} + \int_{0}^{n} \frac{P(y)}{s + y} dy$$

$$= -(s - 1/2)\log(s) + \frac{1}{2}\log(s + n) + (s + n)\log(s + n) + \int_{0}^{n} \frac{P(y)}{s + y} dy.$$
For notational convenience, denote the integral on the right in (1) by $I_n(s)$. Then setting $s = 1$ in (1) and subtracting gives

$$
\sum_{i=0}^{n} \log \left( \frac{s + i}{1 + i} \right) = -(s - 1/2) \log(s) + \frac{1}{2} \log \left( \frac{s + n}{n + 1} \right) + (s-1) \log(s+n) + (n+1) \log \left( \frac{s + n}{1 + n} \right) + I_n(s) - I_n(1).
$$

We now use the identity (exercise 14a)

$$
\Gamma(s) = \lim_{n \to \infty} \frac{(n+1)!}{s(s+1) \cdots (s+n)} n^{s-1}
$$

and take logarithms to get

$$
- \log(\Gamma(s)) = -(s - 1/2) \log(s) + \lim_{n \to \infty} \left[ \frac{1}{2} \log \left( \frac{s + n}{n + 1} \right) + (s-1) \log \left( \frac{s + n}{n} \right) \right]
$$

$$
+ \lim_{n \to \infty} \left[ (n+1) \log \left( \frac{s + n}{1 + n} \right) + I_n(s) - I_n(1) \right]
$$

$$
= -(s - 1/2) \log(s) + \lim_{n \to \infty} \left[ (n+1) \log \left( \frac{s + n}{1 + n} \right) + I_n(s) - I_n(1) \right].
$$

Setting $z = (n + 1)^{-1}$ and then $u = (s-1)z$, we see that

$$
\lim_{n \to \infty} (n+1) \log \left( \frac{s + n}{1 + n} \right) = \lim_{n \to \infty} (n+1) \log \left( 1 + \frac{s - 1}{1 + n} \right)
$$

$$
= \lim_{z \to 0} z^{-1} \log(1 + (s - 1)z)
$$

$$
= \lim_{u \to 0} (s - 1)u^{-1} \log(1 + u)
$$

$$
= s - 1.
$$

Therefore

$$
(2) \quad \log(\Gamma(s)) = (s - 1/2) \log(s) - s + 1 - \lim_{n \to \infty} (I_n(s) - I_n(1)).
$$

Consider the function

$$
g(y) = \int_{0}^{y} P(t) \, dt,
$$

so that $g'(y) = P(y)$. From the definition of $P(y)$,

$$
(3) \quad |g(y)| = \left| \int_{0}^{y} P(t) \, dt \right| \leq \begin{cases} 
0 & \text{if } y \in \mathbb{Z}, \\
1/2 & \text{for all } y \in \mathbb{R}.
\end{cases}
$$

Now integrating by parts yields

$$
I_n(s) = \int_{0}^{n} \frac{P(y)}{s + y} \, dy
$$

$$
= g(y) \bigg|_{y=0}^{n} + \int_{0}^{n} \frac{g(y)}{(s+y)^2} \, dy
$$

$$
= \int_{0}^{n} \frac{g(y)}{(s+y)^2} \, dy,
$$

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so that

\[
\lim_{n \to \infty} I_n(s) = \int_0^\infty \frac{g(y)}{(s+y)^2} \, dy := I(s),
\]

where the integral converges absolutely by (3) (since \(s\) can’t be a negative real number).

We claim that \(|I(s)| \ll 1/|s|\) for \(s \in R(\delta)\), where the implicit constant depends only on \(\delta\). To see why, write \(s = re^{i\theta}\). Then by (3) (with \(rx = y\))

\[
|I(s)| \leq \int_0^\infty \frac{|g(y)|}{|s+y|^2} \, dy
\]

\[
\leq \frac{1}{2r} \int_0^\infty \frac{1}{r|e^{i\theta} + y/r|^2} \, dy
\]

\[
= \frac{1}{2r} \int_0^\infty \frac{dx}{|e^{i\theta} + x|^2}
\]

\[
= \frac{1}{2r} \int_0^\infty \frac{dx}{x^2 + 2x \cos \theta + 1}
\]

\[
\leq \frac{1}{2r} \int_0^\infty \frac{dx}{x^2 - 2x \cos \delta + 1},
\]

since \(\theta = \text{Arg}(s)\) satisfies \(-\pi + \delta \leq \theta \leq \pi - \delta\) by hypothesis. Note that the integrand on the right here is always finite (the denominator is never zero), and for \(x\) sufficiently large it is no more than \(2/x^2\). In particular, that integral is \(O(1)\) (depending only on \(\delta\)), proving our claim.

Combining our claim with (2) and (4), we get after exponentiating

\[
\Gamma(s) \frac{\Gamma(s+1/2) e^{-s}}{s - 1/2} C = \exp \left( -I(s) \right) = 1 + \sum_{m \geq 1} \frac{(-I(s))^m}{m!} = 1 + O(1/|s|),
\]

where \(C = \exp \left( 1 + I(1) \right)\). It remains to evaluate \(C\). To do that, we appeal to the formula (exercise 16b)

\[
\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s).
\]

We take \(s = \sigma \in \mathbb{R}\) and let \(\sigma \to \infty\) in (5) applied to these three Gamma values. We get (after dividing by \(C\), which is certainly not zero)

\[
\lim_{\sigma \to \infty} \sigma^{-1/2} e^{-\sigma} (\sigma + 1/2)^\sigma e^{-(\sigma+1/2)} C = \lim_{\sigma \to \infty} \sqrt{\pi}2^{1-2\sigma} (2\sigma)^{2\sigma-1/2} e^{-2\sigma},
\]

\[
\lim_{\sigma \to \infty} \sigma^{-1/2} (\sigma + 1/2)^\sigma e^{-1/2} C = \lim_{\sigma \to \infty} \sqrt{2\pi} \sigma^{-1/2} \sigma^{2\sigma} e^{-2\sigma},
\]

and thus

\[
C = \lim_{\sigma \to \infty} \sqrt{2\pi} e \left( \frac{\sigma}{\sigma + 1/2} \right)^\sigma
\]

\[
= \sqrt{2\pi}
\]

via a standard limit computed in calculus.

**Lemma 2:** Let \(\delta > 0\) and \(R(\delta)\) be as above. Then for all \(s \in R(\delta)\) with \(|s| \geq 1\) we have

\[
\frac{\Gamma'(s)}{\Gamma(s)} = \log s + O(1/|s|).
\]
Proof: Recall Euler's constant $\gamma$ satisfies

$$\gamma = \sum_{n=1}^{N} \frac{1}{n} - \log N + O(1/N).$$

Using this and the definition of the Gamma function, for integers $N > |s|$

$$\frac{\Gamma'(s)}{\Gamma(s)} = \frac{-1}{s} - \gamma - \sum_{n \geq 1} \left( \frac{1}{s+n} - \frac{1}{n} \right)$$

$$= \frac{-1}{s} - \gamma - \sum_{n=1}^{N} \left( \frac{1}{s+n} - \frac{1}{n} \right) - \sum_{n>N} \left( \frac{1}{s+n} - \frac{1}{n} \right)$$

$$= \frac{-1}{s} + \log N - \sum_{n=1}^{N} \frac{1}{s+n} + \sum_{n>N} \frac{s}{(s+n)n} + O(1/N)$$

$$= \log N - \sum_{n=1}^{N} \frac{1}{s+n} + O(|s|/N).$$

Now by Lemma 1 with $f(x) = 1/(s+x)$ we have

$$\sum_{n=1}^{N} \frac{1}{s+n} = \int_{0}^{N} \frac{1}{s+y} \, dy + \frac{1}{2s} + \frac{1}{2(s+N)} - I_{N}(s)$$

$$= \log(N+s) - \log s + \frac{1}{2s} + \frac{1}{2(s+N)} - I_{N}(s).$$

Therefore

$$\frac{\Gamma'(s)}{\Gamma(s)} = \log N - \log(N+s) + \log s - \frac{1}{2s} - \frac{1}{2(s+N)} + I_{N}(s) + O(|s|/N).$$

Letting $N \to \infty$ yields the desired inequality.