Throughout this homework set, $K$ is a function field with field of constants $\mathbb{F}_q$. All other notation is exactly as in the notes and what we’ve been using in class.

1. Prove that the zeta function $\zeta_K(s)$ has an Euler product:

$$\zeta_K(s) = \prod_{v \in M(K)} (1 - q^{-s \deg(v)})^{-1}$$

valid for all $s$ with $\Re(s) > 1$.

2. Let $w \in M(\mathbb{F}_q(X))$ be a place of the field of rational functions and let $v \in M(K)$ be a place of $K$ lying above $w$, i.e., $v | w$. We showed in class that

$$\sum_{\substack{v \in M(K) \\ v | w}} e_v f_v = [K : \mathbb{F}_q(X)]$$

where the ramification indices $e_v$ are given by $\text{ord}_v(\alpha) = e_v \text{ord}_w(\alpha)$ for all $\alpha \in \mathbb{F}_q(X)$ and the residue class degrees by $f_v \deg(w) = \deg(v)$. Use this and the Euler product above to show that

$$\zeta_K(s) \leq \zeta_{\mathbb{F}_q(X)}(s)^{[K : \mathbb{F}_q(X)]}$$

for all $s > 1$.

3. Let $a_n$ denote the number of effective divisors $\mathfrak{a} \in \text{Div}(K)$ with $\deg(\mathfrak{a}) = n$, so that

$$\zeta_K(s) = \sum_{n \geq 0} a_n q^{-sn}.$$ 

a) Prove that

$$\zeta_{\mathbb{F}_q(X)}(1 + \epsilon) \ll 1$$

for all $\epsilon > 0$, where the implicit constant depends only on $q$ and $\epsilon$.

b) For all integers $m \geq 0$, all real $s \leq 1$, and all $\epsilon > 0$, prove that

$$\sum_{n=0}^{m} a_n q^{-sn} \ll q^{m(1-s+\epsilon)},$$

and for all $s > 1 + \epsilon$

$$\sum_{n \geq m} a_n q^{-sn} \ll q^{-m(s-1-\epsilon)},$$

where the implicit constants depend only on $q$, $[K : \mathbb{F}_q(X)]$, and $\epsilon$. In particular, $a_m \ll q^{m(1+\epsilon)}$ for all $m \geq 0$. (Use #2 above and part a.)

c) By considering the case $m = 2g - 1$ in part b), show that

$$J_K \ll q^{g(1+\epsilon)}$$
for all $\epsilon > 0$, where the implicit constant depends only on $q$, $[K: \mathbb{F}_q(X)]$, and $\epsilon$.

4. We showed in class that

$$
\zeta_K(s) = \sum_{n=0}^{2g-2} a_n q^{-ns} + \frac{J_K q^{s(1-2g)}}{q-1} \left( \frac{q^g}{1-q^{1-s}} - \frac{1}{1-q^{-s}} \right)
$$

for all $s$ with $\Re(s) > 1$. This may be used to analytically continue (define) the zeta function for all complex $s$ except for simple poles at $s = 1$ and $s = 0$. Further, the “Riemann Hypothesis” (Hasse-Weil Theorem) states that this analytically continued zeta function has $2g$ zeros (counted with multiplicity), all of which have real part $1/2$.

a) Prove that the analytically continued zeta function is negative for all real $s$ between $1/2$ and $1$.
b) Let $0 < \epsilon < 1/2$. Assuming that $g \geq 1$, show that $\sum_{n=0}^{2g-2} a_n q^{(\epsilon-1)n} \geq 1$, so that

$$
\frac{J_K q^{(1-\epsilon)(1-2g)}}{q-1} \left( \frac{q^g}{1-q^{\epsilon}} - \frac{1}{1-q^{\epsilon-1}} \right) < -1.
$$
c) Multiply both sides of the inequality in part b) by $(1-q^{\epsilon})(1-q^{\epsilon-1})$ and prove that

$$
J_K \gg q^{g(1-\epsilon)}.
$$