PART A Work 7 of the following 8 problems. Each problem will be given equal weight.

Part A

A1 Prove that the matrix
\[
\begin{pmatrix}
-9 & 4 \\
-25 & 11 \\
\end{pmatrix}
\]
is not diagonalizable.

A2 Let $V$ be a finite dimensional vector space over a field $k$, and let $T$ and $U$ be linear transformations from $V$ to $V$. Assume that $U$ is invertible. Prove that $T$ is diagonalizable if and only if $U^{-1}TU$ is diagonalizable.

A3 Prove that there is no simple group of order 80. You may use Sylow’s theorem.

A4 Let $G$ be a group whose order is a power of the prime $p$. Prove that the center of $G$ has more than one element.

A5 Let $R$ be an integral domain, and let $I$ be a nonzero principal ideal in $R$. Prove that $R^\times$ (the unit group of $R$) acts transitively on the set
\[
\{ r \in R \mid (r) = I \}.
\]

A6 Prove that the polynomial $x^4 + 4x^3 + 6x^2 + 2x + 1$ is irreducible in $\mathbb{Q}[x]$.

A7 Prove that the polynomial $x^{p^n} - x \in \mathbb{F}_p[x]$ is separable. (Here, $n \in \mathbb{N}$.)

A8 Let $E/K$ be an extension of fields, and let $f(x) \in K[x]$ have degree $d$. Prove that $f(x)$ has at most $d$ roots in $E$. 

\[1\]
PART B  Work 3 of the following 4 problems. All problems will be given equal weight.

Part B

B1 Let $E/K$ and $L/E$ be algebraic extensions of fields. Prove that $L/K$ is an algebraic extension of fields.

B2 Let $E_1$ and $E_2$ be Galois extensions of the field $K$, and assume that $L$ is a field containing both $E_1$ and $E_2$. Prove that the compositum $E_1E_2$ is a Galois extension of $K$.

B3 Let $\alpha \in \mathbb{C}$ satisfy the following conditions:

1. The element $\alpha$ satisfies a polynomial in $\mathbb{Q}[x]$ of degree 5.
2. The extension $\mathbb{Q}(\alpha)/\mathbb{Q}$ has degree 120.

Prove that $\mathbb{Q}(\alpha)/\mathbb{Q}$ is Galois and contains precisely one intermediate field $K$ which is Galois over $\mathbb{Q}$. Find $|K : \mathbb{Q}|$.

B4 Let $k$ be a field, and consider the ring

$$A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in k \right\}.$$ 

Find a composition series for the left-regular $A$-module $AA$. How many isomorphism classes of simple left $A$-modules are there?