Ph.D. Qualifying Examination A
Algebra
January 2016

Instructions: For the two-hour examination, work Part A only. For the three-hour examination, work Part A and Part B.

Part A  Solve seven of the following eight problems.

1. Let $G$ be a finite group, let $p$ be a prime, and let $K$ be a normal subgroup of $G$ of index $p$. Prove that, for all subgroups $H$ of $G$, either
   (a) $H \leq K$, or
   (b) $G = HK$ and $|H : H \cap K| = p$.

2. Prove that there is no simple group of order 56.

3. Prove that if $G$ is a (not necessarily finite) group in which the number of elements of order two is exactly three, then $G$ is not simple.

4. Let $F$ denote the field $\mathbb{Z}_2$, and let $R = F[x]/(x^2 + 1)$. Prove that $R$ contains exactly four elements and that $R \not\cong \mathbb{Z}_4$ and $R \not\cong \mathbb{Z}_2 \times \mathbb{Z}_2$, as rings.

5. Let $F/K$ be a field extension, and suppose that $u \in F$ is algebraic over $K$.
   (a) Prove that there exists a unique monic irreducible polynomial $p(x) \in K[x]$ with $p(u) = 0$.
   (b) Prove that if $f(x)$ is any polynomial in $K[x]$ with $f(u) = 0$, then $p(x)$ divides $f(x)$ in $K[x]$.

6. Let $f(x) = x^4 - 4$.
   (a) Determine a splitting field $F$ of the polynomial $f(x)$ over $\mathbb{Q}$.
   (b) Determine the isomorphism type of the Galois group $\text{Aut}(F/\mathbb{Q})$.
   (c) Determine all subgroups of $\text{Aut}(F/\mathbb{Q})$ and their fixed fields.

7. Explain how to construct a field $K$ with 27 elements. What are the subfields of this field?

8. Let $V$ be a finite-dimensional complex vector space, and let $S, T$ be linear operators on $V$ such that $ST = TS$. Recall that a subspace $W$ of $V$ is said to be invariant under $T$ if $T(W) \subseteq W$.
   (a) Prove that if $\lambda$ is an eigenvalue of $S$, then the eigenspace $V_\lambda = \{ \vec{x} \in V \mid S(\vec{x}) = \lambda \vec{x} \}$ is invariant under $T$.
   (b) Prove that $S$ and $T$ have at least one common eigenvector (not necessarily associated to the same eigenvalue).
Part B  Solve three of the following four problems.

1. Prove that, in a principal ideal domain, every nonzero prime ideal is maximal.

2. Let $R$ be a ring, let $M$ be an $R$-module, and let $\phi : M \to M$ be an $R$-module homomorphism.
   (a) Prove that if $M$ is noetherian and $\phi$ is surjective, then $\phi$ is injective.
   (b) Prove that if $M$ is artinian and $\phi$ is injective, then $\phi$ is surjective.

3. Let $R$ be a commutative ring. Prove that $R$ is semisimple if and only if $R$ is isomorphic to a finite direct product of fields.

4. Let $R$ be a commutative ring, and let $I$ be an ideal of $R$. Prove that $I$ is primary if and only if $R/I$ is a nonzero ring with the property that every zero divisor is nilpotent.