Part A  Solve seven of the following eight problems.

1. Let $G$ be a group, and let $H$ and $K$ be isomorphic subgroups of $G$. Prove or disprove:
   \textit{If $H$ is normal in $G$, then $K$ is normal in $G$.}

2. Prove that there is no simple group of order 36.

3. Let $p$ be a prime number, and let $G$ be a nontrivial finite $p$-group.
   \begin{enumerate}
   \item Let $X$ be a finite set on which $G$ acts. Prove that $|X| \equiv |X^G| \pmod{p}$, where $X^G = \{x \in X \mid a \cdot x = x \text{ for all } a \in G\}$.
   \item Use part (a) to deduce that the center of $G$ is nontrivial.
   \end{enumerate}

4. Consider the ring $R = M_2(\mathbb{Z})$ consisting of all $2 \times 2$ matrices with integer entries. Let $I$ be the subset of $R$ consisting of all matrices with even entries. Then $I$ is an ideal of $R$ and the factor ring $R/I$ is finite. \textbf{Your task:} Determine the exact number of elements contained in $R/I$.

5. Let $F/K$ be a field extension, and suppose that $u \in F$ is algebraic over $K$.
   \begin{enumerate}
   \item Prove that there exists a unique monic irreducible polynomial $p(x) \in K[x]$ with $p(u) = 0$.
   \item Prove that if $f(x)$ is any polynomial in $K[x]$ with $f(u) = 0$, then $p(x)$ divides $f(x)$ in $K[x]$.
   \end{enumerate}

6. Determine the isomorphism type of the Galois group of the polynomial $f(x) = x^4 - 2$ over $\mathbb{Q}(i)$.

7. Let $F/K$ be a Galois extension, and suppose that $\text{Aut}(F/K) \cong S_4$.
   \begin{enumerate}
   \item Prove that there are at least 10 distinct fields strictly between $K$ and $F$.
   \item Prove that there is a field $E$ strictly between $F$ and $K$ such that $E/K$ is Galois. Describe the Galois group of $E/K$.
   \end{enumerate}

8. Let $V$ be a finite-dimensional complex vector space, and let $S$, $T$ be linear operators on $V$ such that $ST = TS$. Recall that a subspace $W$ of $V$ is said to be invariant under $T$ if $T(W) \subseteq W$.
   \begin{enumerate}
   \item Prove that if $\lambda$ is an eigenvalue of $S$, then the eigenspace $V_\lambda = \{\vec{x} \in V \mid S(\vec{x}) = \lambda \vec{x}\}$ is invariant under $T$.
   \item Prove that $S$ and $T$ have at least one common eigenvector (not necessarily associated to the same eigenvalue).
Part B  Solve three of the following four problems.

1. Prove that, in a principal ideal domain, every nonzero prime ideal is maximal.

2. Let $R$ be a ring, and let $M$ be an $R$-module. Prove that $M$ is artinian (that is, $M$ satisfies the descending chain condition) if and only if every nonempty set of $R$-submodules of $M$ has a minimal element.

3. Let $R$ be a commutative ring. Prove that $R$ is semisimple if and only if $R$ is isomorphic to a finite direct product of fields.

4. Let $T$ be an integral ring extension of $S$, and let $S$ be an integral ring extension of $R$. Prove that $T$ is an integral ring extension of $R$. 