Instructions: Write your answers to problems A1-A5, your answers to problems B1-B5, and your answers to problem B6-B10 in separate blue books. All candidates should attempt 4 of the 5 problems in Part A. Those taking the 2-hour examination should work 4 of the 10 problems in Part B. Those taking the 3-hour examination should work 8 of the 10 questions in part B. Clearly indicate which problems you wish to have scored.

PART A: Work 4 of the following 5 problems. Clearly indicate which problem is not to be graded.

A1. Let \( f : [0, 1] \rightarrow \mathbb{R} \) be defined by \( f(x) = \begin{cases} 
\frac{x \cos \left( \frac{1}{x} \right)}{x}, & x \in (0, 1] \\
0, & x = 0 
\end{cases} \)

(a) Show that \( f \) is continuous at \( x = 0 \)

(b) Show that \( f \) is uniformly continuous on \([0, 1]\)

(c) Is \( f \) differentiable at \( x = 0 \)? Explain.

A2. Define \( f : \mathbb{R} \rightarrow \mathbb{R} \) as follows.

\[
f(x) = \begin{cases} 
8x & \text{if } x \text{ is a rational number} \\
2x^2 & \text{if } x \text{ is not a rational number} 
\end{cases}
\]

At what values of \( x \) does \( f \) have a limit? Is it continuous there? Is it differentiable there?

A3. State the Mean Value Theorem. Hence or otherwise prove that \(|\sin(x)| < |x|\) for \( x \in [-\pi/2, \pi/2] \)
A4. Compute the following limits if they exist

(a) \( \lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + y^2} \)

(b) \( \lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2} \)

A5. State the formal definition, in terms of \( \epsilon \) and \( N \), for a series to converge.

Which of the following series converge? Explain your answers

(a) \( \sum \frac{n^2}{n^2 + 1} \)

(b) \( \sum \frac{1}{n \ln(n)} \)

(c) \( \sum \frac{7^n+3}{11^n} \)

(d) \( \sum \frac{1}{n^2 + 4n - 1} \)

PART B

B1.

1. (a) Determine the limit for the sequence \( \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} dx \), and prove your result.

(b) If \( \{f_n\} \) is a sequence of continuous functions on \([0,1]\) such that \( 0 \leq f_n \leq 1 \) and \( \lim_{n \to \infty} f_n(x) = 0 \) for every \( x \) in \([0,1]\), prove

\[ \lim_{n \to \infty} \int_0^1 f_n(x) dx = 0. \]

B2. (a) Define the Cantor ternary set \( C \) on the interval \([0,1]\).

(b) Prove \( C \) is closed.

(c) If \( \sim C \) denotes the complement of \( C \) in \([0,1]\), prove \( m(\sim C) = 1 \), where \( m \) denotes Lebesgue measure.
B3. Let \( f \) be an absolutely continuous, nondecreasing function on a finite, closed interval \([a, b]\). Let \( E \) be a set of Lebesgue measure zero in \([a, b]\). Prove that the measurable set \( f(E) \) has Lebesgue measure zero.

B4. Let \( M \) be the collection of Lebesgue measurable sets in \( \mathbb{R} \). A set \( A \in M \) is called an atom if \( m(A) > 0 \), and for all measurable sets \( B \subseteq A, m(B) = 0 \) or \( m(A \setminus B) = 0 \). Prove no atoms exist if \( m \) denotes Lebesgue measure.

B5. Let \( \{f_n\} \) be a sequence of finite a.e. Lebesgue measurable functions on a set \( X \subseteq \mathbb{R} \). We say \( \{f_n\} \) converges almost uniformly to \( f \) on \( X \), and write \( f_n \to f[a.u.] \), if for every \( \epsilon > 0 \) there exists a measurable set \( E \subseteq X \) such that \( m(X \setminus E) < \epsilon \) and \( \{f_n\} \) converges uniformly to \( f \) on \( E \).

(a) State carefully and completely Egoroff’s theorem.

(b) Prove that if \( f_n \to f[a.u.] \) on \( X \), then \( f_n \to f(a.e.) \) on \( X \) and \( f_n \to f[meas] \) on \( X \).

B6. Can \( \Omega = \{z; \ 1/2 < |z| < 1, \ \arg z \neq 0\} \) be mapped conformally onto a rectangle? If so, determine the mapping.
B7. Let $f$ and $g$ be analytic in a region $G$. If $\text{Re}f = \text{Re}g$ in $G$, then prove

$$f(z) = g(z) + ic, \quad \text{for all} \quad z \in G,$$

where $c$ is a real constant.

B8. Suppose that $G$ is a bounded region, and let $f_n$ be continuous in $\overline{G}$ and analytic in $G$ for each $n \in \mathbb{Z}^+$. Prove that if $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent on $\partial G$, then $\sum_{n=1}^{\infty} f_n(z)$ is uniformly convergent on $G$, where $\partial G$ is the boundary of $G$.

B9. Find the number of zeros of $z^8 - 4z^5 + z^2 - 1 = 0$ in the unit disk $B(0, 1) = \{z : |z| < 1\}$.

B10. Let $f$ be an entire function and $n$ an integer. If

$$|f(z)| \leq 4|z|^n \quad \text{for} \quad |z| \geq 100,$$

show that $f$ is a polynomial of degree at most $n$. 