Instructions: Write your answers to problems A1-A5, your answers to Problems B1-B5, and your answers to problems B6-B10 in separate blue books. All candidates should attempt 4 of the 5 problems in part A. Those taking the two hour examination should work 4 of the 10 problems in Part B. Those taking the three hour examination should work 8 of the 10 problems in part B. Clearly indicate which problems you wish to have scored.

Part A: Work 4 of the following 5 problems. Clearly indicate which problem is not to be graded.

A1. Suppose that \((a_n)_{n=1}^{\infty}\) and \((b_n)_{n=1}^{\infty}\) are sequences.

(a) If \(a_n \to 0\) and \((b_n)_{n=1}^{\infty}\) is bounded, prove that the sequence defined by \(c_n = a_n b_n\), for \(n \in \mathbb{N}\), converges to 0 as \(n \to \infty\).

(b) Hence, or otherwise, find the limit of the sequence given by \(d_n = \frac{n + 2 \cos(n)}{3 \sin(n) + 4n}\).

A2. Consider the function \(f(x) = \begin{cases} x^2 + x^3 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}\).

(a) Determine where \(f\) is continuous.

(b) Determine where \(f\) is differentiable.

Fully explain your answers.

A3. For \(n \in \mathbb{N}\) and \(n \geq 2\), consider the functions \(f_n(x) = \begin{cases} 1 & \text{if } x \in [0, 1/n^2] \\ \frac{n^2 x - n}{1 - n} & \text{if } x \in (1/n^2, 1/n) \\ 0 & \text{if } x \in [1/n, 1] \end{cases}\).

(a) Sketch the graph of \(y = f_n(x)\).

(b) Prove that the sequence \((f_n)_{n=1}^{\infty}\) converges pointwise on \([0, 1]\) and find the limit.

(c) Prove that \((f_n)_{n=1}^{\infty}\) does not converge uniformly on \([0, 1]\).

A4. Consider series \(\sum_{n=1}^{\infty} a_n\) of non-negative terms.

(a) State the Comparison Test and the Ratio Test for such series.

(b) Using these two tests, determine the convergence or divergence of \(\sum_{n=1}^{\infty} \frac{e^n - n^2/10}{(n + 1)! \log(1 + 2n)}\).
A5. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a function.

(a) Define what it means for $F$ to be differentiable at $X_0 \in \mathbb{R}^n$.

(b) Find $F'(X_0)$ for

$$F(x, y, z) = \begin{pmatrix} \exp(x^2 - y + \sin(z)) \\ ye^x - 2z^2 \end{pmatrix}, \quad X_0 = (1, 2, \pi/2).$$

(c) Hence write down the affine map that well approximates $F$ near $X_0$. 
Part B.

B1. (a) State the Lebesgue Convergence Theorem.

(b) Find \( \lim_{n \to \infty} \int_1^\infty \frac{2n}{e^{2x} + n^2 x} \, dx \) and justify your calculations.

B2. Let \( E \) be a bounded set of real numbers.

(a) Prove that there is a \( G_\delta \) set \( G \) such that \( E \subset G \), and \( m(G) = m^*(E) \).

(b) Let \( G \) be a \( G_\delta \) set in (a). If \( m^*(G \setminus E) = 0 \), prove that \( E \) is measurable.

B3. Prove that \( f \) is of bounded variation on \([a, b]\) if and only if there are increasing function \( g \) and \( h \) on \([a, b]\) such that \( f = g - h \).

B4. Let \( E \subset [a, b] \) be a measurable set with \( m(E) > 0 \).

(a) For \( a \leq x \leq b \), define \( h(x) = m(E \cap [a, x]) \). Prove that \( h \) is a continuous function on \([a, b]\).

(b) Prove that \( \lim_{h \to 0} \frac{m(E \cap [x, x+h])}{h} = \chi_E(x) \) a.e. on \([a, b]\).

B5. Let \( \langle f_n \rangle \) be a sequence of integrable functions such that \( f_n \to f \) with \( f \) integrable.

Suppose that \( \lim_{n \to \infty} \int |f_n(x)| \, dx = \int |f(x)| \, dx \). Prove that

\[
\lim_{n \to \infty} \int |f(x) - f_n(x)| \, dx = 0.
\]
B6. Let $G = \{ z : |z| < 1 \text{ and } \text{Re} z > 0 \}$, $H = \{ z : \text{Im} z > 0 \text{ and } \text{Re} z > 0 \}$ and $S = \{ z : |z| < 1 \}$.

(a) Find a conformal mapping from $G$ onto $H$.
(b) Find a conformal mapping from $H$ onto $S$.

B7. Evaluate $\int_{\Gamma} \frac{1}{(z + 4)^3(z - 2i)} \, dz$, where:

a) $\Gamma$ is the circle $|z| = 1$ traversed once counterclockwise.

b) $\Gamma$ is the circle $|z| = 3$ traversed once counterclockwise.

c) $\Gamma$ is the circle $|z| = 5$ traversed once counterclockwise.

B8. Let $f$ be an entire function in $\mathbb{C}$. If $f(z)$ is real when $|z| = 1$, prove that $f$ is constant in $\mathbb{C}$.

B9. (a) State one of any version of Residue Theorem.

(b) Let $f$ be an entire function and let $a_1, \ldots, a_n$ be all zeros of $f$ in $\mathbb{C}$. Suppose that there exist real numbers $r_0 > 0$ and $t > 1$ such that $|f(z)| \geq |z|^t$ for all $|z| \geq r_0$. Prove that

$$\sum_{j=1}^{n} \text{Res}(\frac{1}{f}, a_j) = 0.$$ 

B10. (a) State one of any version of Rouche’s Theorem.

(b) Determine the number of solutions of the equation $z^{10} + 10z + 8 = 0$ in the unit disk $|z| < 1$. 