Instructions: In this three-hour examination, Part A and Part B carry equal weight in determining your overall performance. Please use separate blue books for Part A and Part B.

Answer all 4 questions in Part A and 4/5 questions in Part B.
PART A

A1. Let $f(t,x) \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, and that $f$ satisfies a local Lipschitz condition in $x$. Assume $f(t,0) = 0$. If $x(t)$ is a solution of the equation $\dot{x} = f(t,x)$ such that $x(0) \neq 0$, show that $x(t) \neq 0$ for any $t \in \mathbb{R}$. 

A2. Show that for some $\mu \neq 0$, the system

$$\begin{align*}
\dot{x} &= \mu x - y + xy^2 - x^3 \\
\dot{y} &= x + \mu y - x^2 - y^3
\end{align*}$$

has a nonconstant periodic orbit.

A3. Consider the system $\dot{x} = Ax$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$ 

Let $x(t)$ be a solution of the above system. We say $x(t)$ grows linearly if $\lim_{t \to \infty} \frac{|x(t)|}{t} = c > 0$, and grows superlinearly (faster than linearly) if $\lim_{t \to \infty} \frac{|x(t)|}{t} = \infty$. Find all initial conditions $x(0)$ such that the respective solutions $x(t)$

(a) Are bounded.
(b) Grow linearly.
(c) Grow superlinearly.

A4. Consider

$$\ddot{x} + \alpha \dot{x} + g(x) = 0 \quad (1)$$

with $\alpha > 0$, $g$ a $C^1$ function with $xg(x) > 0$ for $x \neq 0$, $\int_{-\infty}^{0} g(x) \, dx = \infty$, and $\int_{0}^{\infty} g(x) \, dx = c < \infty$. Prove that every bounded solution $x(t)$ of equation (1) satisfies

$$\lim_{t \to \infty} x(t) = \lim_{t \to \infty} \dot{x}(t) = 0$$

and that every solution $x(t)$ is bounded on $t \in (0, \infty)$. 

1
B1. (i) Let \( \Omega \) be a domain in \( \mathbb{R}^N, N \geq 1 \), given \( u \in L^p(\Omega), p \geq 1 \), define \( \frac{\partial u}{\partial x_i} \), \( 1 \leq i \leq N \), the weak derivatives of first order for \( u \) and hence, the Sobolev Space \( W^{1,p}(\Omega) \).

(ii) For \( u(x) = \frac{1}{|x|^\alpha}, \alpha > 0 \), what are the values for \( \alpha \) and \( p \) in terms of \( N \), would \( u \) belongs to \( W^{1,p}(\Omega) \)? If you wish, you may identify \( \Omega \) with \( B(0,1) \), the unit ball in \( \mathbb{R}^N \).

B2. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^N, N \geq 1 \) which has a \( C^1 \) boundary \( \partial \Omega \) and consider the Poission Dirichlet problem,

\[
\begin{aligned}
\Delta u &= f & \text{in } \Omega, f &\in L^p(\Omega), p \geq 1 \\
\frac{\partial u}{\partial n} &= 0 & \text{on } \partial \Omega &\text{ in the sense of trace.}
\end{aligned}
\]

Suppose we formulate the solution by seeking \( u \in W^{2,p}_0(\Omega) \) such that \(-\Delta u = f\), criticize this formulation and if you don’t agree with the formulation, how would you reformulate it? Explain your answer.

B3. Consider the non-homogeneous Poission Dirichlet problem,

\[
\begin{aligned}
\Delta u &= f & \text{in } B(0;R), & \text{the ball with radius } R \text{ in } \mathbb{R}^N \text{with} N \geq 2, \\
\frac{\partial u}{\partial n} &= g & \text{on } \partial B(0;R)
\end{aligned}
\]

Prove that, for its solution \( u \), we have the mean-value formula,

\[
u(0) = \frac{1}{\omega_N R^{N-1}} \int_{\partial B(0, R)} g(\xi) dS_\xi + \frac{1}{(N-2)\omega_N} \int_{B(0, R)} \left( \frac{1}{R^{N-2}} - \frac{1}{|\xi|^{N-2}} \right) f(\xi) d\xi.
\]

where \( \frac{1}{\omega_N R^{N-1}} \int_{\partial B(0, R)} g(\xi) dS_\xi \) being the mean value of \( g \) over the sphere \( \partial B(0, R) \),

and \( \omega_N \) is the area of the spherical surface of radius 1 in \( \mathbb{R}^N \).

B4. Consider the PDE of the Neuman type,

\[
\begin{aligned}
\Delta u &= 1 & \text{in } \Omega, \\
\frac{\partial u}{\partial n} &= 0 & \text{on } \partial \Omega,
\end{aligned}
\]

Where \( n \) is the exterior unit normal vector at \( \partial \Omega \), prove that this problem has no solution.
B5. (i) Solve the 1-D Wave Cauchy Problem

\[
\begin{align*}
&\begin{cases}
  \frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0, \quad -\infty < x < \infty, \ t > 0, \\
  u(x, 0) = \sin x, \quad -\infty < x < \infty, \\
  u_t(x, 0) = \cos x, \quad -\infty < x < \infty.
\end{cases}
\end{align*}
\]

(ii) Consider now the wave equation in the \(\mathbb{R}^N\) setting, \(N \geq 2\),

\[
\begin{align*}
&\begin{cases}
  \frac{\partial^2 u}{\partial t^2} - c^2 \Delta u = 0 \quad \text{in} \quad \Omega, \ t > 0, \ c > 0, \\
  u(x, 0) = g(x), u_t(x, 0) = h(x), x \in \Omega, \\
  u(x, t) = f(x), x \in \partial \Omega.
\end{cases}
\end{align*}
\]

Prove that \(u\) is unique (Hint: Let \(u\) and \(v\) both be solutions and set \(w= u-v\), show that \(w=0\)).